Math 618 Assignment 1

Professor:	*Richard Hall*
Instructions:	*Please explain your solutions carefully.*
Due Date:	*5^{th} February 2013.*

1.1 Consider the functional $J[y] = \int_1^2 x^2(y')^2dx$ with BC $y(1) = 2$ and $y(2) = 1$.
  (i) Find the minimum of $J[y_a] = C(a)$ over the family of 1-parameter 'trial functions' given by $y_a(x) = 3 - x + a(x-1)(2-x)$, where a is a variational parameter.
  (ii) Solve this problem exactly by using the Euler-Lagrange equation, and compare the result with the approximate solution found in (i). It might be interesting to compare the approximate and exact minmizing curves, as well as the values obtained for J[y].

1.2 Consider the refraction of a ray of light which crosses a smooth boundary B between two regions of different but constant refractive index, respectively n_1 and n_2. Use Fermat's principle to derive Snell's law $n_1 \sin\theta_1 = n_2 \sin\theta_2$, where θ_i is the angle which the ray i makes with the normal to B, $n_i = c/v_i$, v_i is the speed of light in region i, and c is the speed of light *in vacuo.* Since $v_i \le c$, the refractive index $n_i \ge 1$.

1.3 We may define the *derivative* $D(x)$ of $f(x)$ by the following: the function f is differentiable at x and has derivative $D(x)$ if the difference $f(x+h) - f(x)$ has the representation

$$f(x+h) - f(x) = D(x)h + R(x,h), \tag{1.3a}$$

where the 'error term' $R(x,h)$ satisfies

$$\lim_{h\to 0}\left(\frac{R(x,h)}{h}\right) = 0.$$

Prove that this definition is equivalent to the 'usual' definition, that is to say:

$$D(x) = f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right). \tag{1.3b}$$

Note that your proof of equivalence must have two parts, $\Rightarrow$ and $\Leftarrow$. The linear function $D(x)h$ of the difference h is the *differential* of f at x; the factor $D(x)$ is the derivative.

1.4 Consider the baseball (or garden-hose) problem. A ball of mass m is thrown up from the point $(0,0)$ over a flat Earth with gravitational constant g and negligible air resistance. The position of the ball is given by $\boldsymbol{r}(t) = (x(t), y(t))$, where $t \ge 0$ is the time. If in Hamilton's Principle,

$$\int_{t_1}^{t_2} L dt \to \text{extremum},$$

the Lagrangian is given by $L = T - V$, where $T = \frac{1}{2}m\left(\boldsymbol{r}'(t)\right)^2$ and $V = mgy(t)$, show that the Euler-Lagrange equations for this variational problem lead to Newton's second law in each dimension. That is to say, the functions $\{x(t), y(t)\}$ satisfy the pair of differential equations

$$x''(t) = 0, \quad y''(t) = -g, \tag{1.4a}$$

which for this problem are independent of m. Suppose that the initial conditions are $x(0) = y(0) = 0$, and $x'(0) = x'_0 \ge 0$, $y(0) = y'_0 > 0$. Let the initial angle α be given by $\tan(\alpha) = y'_0/x'_0$, and the initial speed by $u = \sqrt{(x'_0)^2 + (y'_0)^2} > 0$. The trajectory $\boldsymbol{r}(t;\alpha,u)$ of the ball is then parametrized by the two parameters $\{\alpha, u\}$. Find $\boldsymbol{r}(t;\alpha,u)$ and sketch a family of these trajectories with u fixed and a range of values of $\alpha \in [0, \pi/2)$. If $u > 0$ is fixed, find (a) the value of α which maximizes the maximum height reached by the ball, and the corresponding maximum height; and (b) the value of α which maximizes the horizontal distance reached by the ball (that is to say, x when $y = 0$ again). It may help to write the initial velocity in the form

$$\boldsymbol{r}'(0) = (x'(0), y'(0)) = (u\cos(\alpha), u\sin(\alpha)) \quad\Rightarrow\quad ||\boldsymbol{r}'(0)|| = u. \tag{1.4b}$$