Math 618 Assignment 2

Professor:	*Richard Hall*
Instructions:	*Please explain your solutions carefully.*
Due Date:	5^{th} *March 2013*

2.1 Consider a linear function $f : \mathbb{R}^3 \to \mathbb{R}$. Demonstrate Riesz's lemma for this example, namely that f has the representation $f(x) = <a, x>$, where a is a fixed element of $\mathbb{R}^3$ depending on f, and $<x, y>$ is the 'usual' inner (or dot) product between the vectors x and y in $\mathbb{R}^3$. If $f(1, 2, 1) = 9$, $f(2, -1, -2) = 3$, and $f(2, 1, -1) = 6$, find the vector a which characterizes the linear function f in the standard basis $\boldsymbol{e} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

2.2 Use the calculus of variations to find the geodesics on the cone

$$\boldsymbol{r}(r, \theta) = (r\cos\theta,\ r\sin\theta,\ 2 - r), \quad 0 \le r \le 2,\ 0 \le \theta \le \pi.$$

In particular, find the shortest distance between $(1, 0)$ and $(1, \pi/2)$ and verify your answer by flattening the cone and finding the distance between the points in the Euclidean plane. This latter task can be done with the aid of a carefully constructed paper model.

2.3 Discuss the family of extremals for the 'mirage problem' for which the refractive index is given by $n(x, y) = 1 + \alpha y$, where α is small and positive. By studying the extremals from the 'observation point' $P : (x, y) = (0, 1)$ with $\alpha = 0.2$, show that a tree of height 1 at $x = 9$ is invisible to an observer at P. If you wish, you may answer this part of the problem by plotting a family of extremals from P with the aid of a computer; the invisible region is that beyond the envelope of this family.

2.4 Consider the brachistochrone in the case that the end points are $y(0) = 0$ and $y(b) = B$. By a convention y points downwards. This is an 'optical problem' in which $n(x, y) = 1/\sqrt{2gy}$ and the functional $J[y]$ to be minimized is given by:

$$J[y] = \frac{1}{\sqrt{2g}} \int_0^b \frac{\sqrt{1 + (y')^2}}{\sqrt{y}} dx.$$

(i) Find the extremal $\hat{y}$ in units in which $2g = 1$ and $b = B = 1$. Also find the minimum time $J[\hat{y}]$.

(ii) Explore the case $b = B = 1$ by using the trial function

$$y_\alpha(x) = x + \alpha x(1-x), \quad \alpha \in \mathbb{R}.$$

That is to say, find $C(\alpha) = J[y_\alpha]$ and minimize (approximately) with respect to α. Compare the best estimate $C(\hat{\alpha})$ with $J[\hat{y}]$ found in part (ii).

(iii) Now consider a given fixed U-shaped brachistochrone from $(0,0)$ to $(2d,0)$, with a minimum point at (d,M). We now suppose that the bead starts from rest at point x, where $0 < x < d$. Let $T(x)$ be the time it takes for the bead to fall from the start to the bottom at (d,M). Find $T(x)$ explicitly and show that this time is a constant (independent of x). Thus, in this experiment the bead always takes the same time to fall to the centre, no matter from where it starts. Historically, this 'equal-time curve' was called the 'tautochrone'; it was a discovery that it is the same curve as the brachistochrone. If, for example, $d = 1$Km, it is, perhaps, rather surprising that the time it takes for the bead to travel more than 1 Km from $x = 0$ to the bottom of the U is the same as when $d - x$ is, say, only 1 mm.