Math 264 Sec A Midterm Test 28 th October 2010

Professor: *Richard Hall*

Instructions: *Please answer all 3 questions which carry equal marks.*
Duration: 1 hour. Please explain your work clearly.

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1. Consider the curve in $\Re^3$ given by $\boldsymbol{r}(t) = (t\cos(t),\ t\sin(t),\ t)\ ,\ t \geq 0$.

   (a) Give a rough sketch of this curve. HINT: look at $x^2 + y^2$.

   (b) Find the tangent line to the curve which touches the curve at $t = \pi/2$.

   (c) Find the plane that the curve passes through orthogonally at $t = \pi/2$.

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2. Consider the curve given in polar coordinates by $r(\theta) = 1 + \cos(3\theta),\ \theta \in [0, 2\pi]$.

   (a) Give a rough sketch of this curve.

   (b) Find the area inside the curve given in (a).

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3. Consider the function $e(x)$ defined by

$$e(x) = \int_0^x e^{-t^2}\, dt.$$

   (a) Find a Taylor series $T(x)$ about $x = 0$ for $e(x)$.

   (b) What is the interval of convergence of $T(x)$?

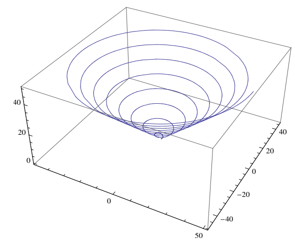
   (c) Find a Taylor polynomial $T_n(x)$ to approximate $e(x)$ with error less than 10^{-6} for $0 \leq x \leq \frac{1}{2}$, that is to say, find n .

   Now use $T_n(x)$ to estimate $e(\frac{1}{2})$.

Solution Notes

1.

(a) Following the hint, we see $x^2 + y^2 = z^2$. That is, the curve lies on a



circular cone with apex at the origin.

(b) A tangent vector at t is given by $\boldsymbol{r}'(t) = (\cos(t) - t\sin(t),\ \sin(t) + t\cos(t),\ 1)$. Thus the equation of the tangent line is

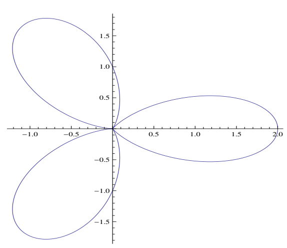
$$\boldsymbol{l}(t) = \boldsymbol{r}(\pi/2) + t\boldsymbol{r}'(\pi/2) = \mathbf{r}_o + t\mathbf{v},$$

where $\mathbf{r}_0 = (0, \pi/2, \pi/2)$ and $\mathbf{v} = (-\pi/2, 1, 1)$.

(c) For the plane, $\mathbf{r}_o$ is a point in the plane, and $\mathbf{v}$ is a normal vector. Hence the equation to the required plane is $(\boldsymbol{r} - \mathbf{r}_o) \cdot \mathbf{v} = 0$.

2

(a) The curve is a rose with three petals: $\boldsymbol{r} = \mathbf{0}$ when $\theta = \pm\{\pi/3,\ \pi\}$.



(b) The area A is given by the integral

$$A = \frac{1}{2}\int_0^{2\pi} (1 + \cos(3\theta))^2\, d\theta = \frac{3\pi}{2}.$$

3

(a) We use the series $e^s = \sum_{k=0}^{\infty} s^k/k!$ with $s = -t^2$, and integrate $\int_0^x (\ldots)dt$, term-by-term, to obtain $e(x) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{k!(2k+1)}(-1)^k$.

(b) The series for $e(x)$, like that for e^x, converges for all $x \in \Re$.

(c) Since the series is alternating, the error incurred in using $T_n(x)$ is less than the maximum absolute value of 'the next term'. We find by exploring that $(\frac{1}{2})^{11}/(5!11) \approx 3.7 \times 10^{-7}$. Hence, the Taylor polynomial $T_9(x)$ is satisfactory, and is given by

$$T_9(x) = x - \frac{x^3}{3} + \frac{x^5}{2!5} - \frac{x^7}{3!7} + \frac{x^9}{4!9}.$$

We find $T_9(\frac{1}{2}) \approx 0.461281364$, whereas $e(\frac{1}{2}) \approx 0.461281006$.
