Math 264 Sec A Midterm Test 27^{th} October 2011

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Instructions: *Please answer all 3 questions which carry equal marks. Duration: 1 hour. Please explain your work clearly.*

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1. Consider the curve in $\Re^3$ given by $\boldsymbol{r}(t) = (t^3 + 3t, t^2 + 1, 3t + 4)$, $t \geq 0$.

   (a) Find the general unit tangent vector $\mathbf{T}(t)$ and its value $\mathbf{T}(1)$.

   (b) Find the tangent line to the curve which touches the curve at $t = 1$.

   (c) Find the principal normal $\mathbf{N}$ and the binormal $\mathbf{B}$ at $t = 1$.

   [HINT: find $\mathbf{B}$ first.]

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2. Consider the curve given in polar coordinates by $r(\theta) = \cos(2\theta)$, $\theta \in [0, 2\pi]$.

   (a) Give a rough sketch of this curve.

   (b) Find the area inside the curve given in (a).

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3. Consider the function $J(x)$ defined by

$$J(x) = \int_0^x \frac{\sin(t^2)}{t^2}\, dt.$$

   (a) Find a Taylor series $T(x)$ about $x = 0$ for $J(x)$.

   (b) What is the interval of convergence of $T(x)$?

   (c) Find a Taylor polynomial $T_n(x)$ to approximate $J(x)$ with error less than 10^{-6} for $0 \leq x \leq 1$.

   (d) Now use $T_n(x)$ to estimate $J(0.9)$.

Solution Notes

1.

(a) We have $\boldsymbol{r}(t) = (t^3 + 3t, t^2 + 1, 3t + 4)$ and

$$\mathbf{T}(t) = \frac{\boldsymbol{r}'(t)}{\|\boldsymbol{r}'(t)\|} = \frac{(3(t^2+1), 2t, 3)}{[9t^4 + 22t^2 + 18]^{\frac{1}{2}}} \rightarrow \mathbf{T}(1) = (6,2,3)/7.$$

(b) A tangent vector at $t = 1$ is given by $\boldsymbol{r}'(t) = (6,2,3)$. Thus the equation of the tangent line is

$$\boldsymbol{l}(t) = (4,2,7) + t(6,2,3).$$

(c) $\mathbf{B}(1) = \boldsymbol{r}'(1) \times \boldsymbol{r}''(1)/\|\boldsymbol{r}'(1) \times \boldsymbol{r}''(1)\| = (-1,3,0)/\sqrt{10}$. Meanwhile $\mathbf{N}(1) = \mathbf{B}(1) \times \mathbf{T}(1) = (9,3,-20)/(7\sqrt{10})$. Note that $\mathbf{N}(1) = \mathbf{T}'(1)/\|\mathbf{T}'(1)\|$ gives the same result, but the calculation takes longer.

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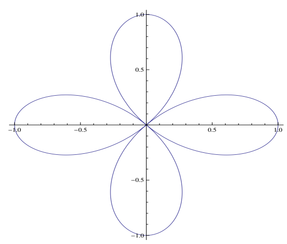
(a) The curve is a rose with four petals: $\boldsymbol{r} = \mathbf{0}$ when $\theta =$



$\{\pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4\}$.

(b) The area A is given by the integral

$$A = \frac{1}{2}\int_0^{2\pi} \cos^2(2\theta)\, d\theta = \frac{\pi}{2}.$$

3

(a) We use the series $\sin(s) = \sum_{k=0}^{\infty} s^{2k+1}(-1)^k/(2k+1)!$ with $s = t^2$, and integrate $\int_0^x (\ldots)dt$, term-by-term, to obtain $J(x) = \sum_{k=0}^{\infty} \frac{x^{4k+1}}{(2k+1)!(4k+1)}(-1)^k$.

(b) The series for $J(x)$, like that for $\sin(x)$, converges for all $x \in \Re$.

(c) Since the series is alternating, the error incurred in using $T_n(x)$ is less than the maximum absolute value of 'the next term'. We find by exploring (for $x = 1$) that $1/(7!11) \approx 1.62 \times 10^{-7}$. Hence, the following Taylor polynomial is satisfactory:

$$T(x) = x - \frac{x^5}{3!5} + \frac{x^9}{5!9} - \frac{x^{13}}{7!13}.$$

We find $T(0.9) \approx 0.8806718431$, whereas $J(0.9) \approx 0.8806718701$.
