Math 264 Sec A Midterm Test 30 th October 2012

Professor:	*Richard Hall*
Instructions:	*Please answer all 3 questions which carry equal marks.*
	Duration: 1 hour. Please explain your work clearly.

1. In a pyramidal molecular model for ammonia NH_3, the hydrogens H are (in suitable units) at the points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, and the nitrogen atom N is at $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ in $\Re^3$.
   (a) Find the distance between N and the plane HHH.
   (b) Find the angle $\theta = \angle HNH$ in degrees between two NH bonds.

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2. Consider the curve in $\Re^3$ given by $\boldsymbol{r}(t) = (a\cos(t), bt, a\sin(t))$, $t \in \Re$, where $a > 0$ and $b > 0$ are constants.
   (a) How would you describe this curve?
   (b) Find expressions for the unit tangent vector $\mathbf{T}(t)$, the principal normal $\mathbf{N}(t)$, the curvature $K(t)$, and the arc length of the segment $t \in [0, 2\pi]$.

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3. Consider the function $\sinh(t) = \frac{1}{2}(e^t - e^{-t})$ and the integral $I(x)$ defined by

$$I(x) = \int_0^x \frac{\sinh(t)}{t}\, dt.$$

   (a) Find Taylor series about $t = 0$ for $\sinh(t)$ and for $f(t)$,
   where $f(t) = \sinh(t)/t$ for $t \neq 0$, and $f(0) = 1$.
   What are the intervals of convergence?
   HINT: you may use the series for the exponential function.
   (b) Find a Taylor polynomial $T_5(x)$ of degree 5 to approximate $I(x)$.
   (c) Use (b) to approximate $I(\frac{1}{2})$.

Solution Notes

1.

(a) The unit normal to HHH is $\mathbf{n} = (1,1,1)/\sqrt{3}$. We project HN onto $\mathbf{n}$ and the distance is $d = |((\frac{1}{2},\frac{1}{2},\frac{1}{2}) - (1,0,0)) \cdot \mathbf{n}| = 1/(2\sqrt{3}) \approx 0.288675$.

(b) We consider the dot product between unit vectors along two NH bonds and find $\theta = \arccos((-\frac{1}{2},\frac{1}{2},\frac{1}{2}) \cdot (\frac{1}{2},-\frac{1}{2},\frac{1}{2})4/3)$,
that is, $\theta = \arccos(-1/3) \approx 1.91063 \equiv 109.471^o$.

2.

(a) The curve is a circular helix about the y-axis.

(b) We have $\boldsymbol{r}'(t) = (-a\ \sin(t), b\,, a\ \cos(t))$ and $\boldsymbol{r}''(t) = (-a\ \cos(t), 0, -a\ \sin(t))$.
Thus $\boldsymbol{r}' \times \boldsymbol{r}'' = a(-b\sin(t), -a, b\cos(t))$, and we find
$K = |\boldsymbol{r}' \times \boldsymbol{r}''|/|\boldsymbol{r}'|^3 = \frac{a}{a^2+b^2}$
$\mathbf{T} = \frac{\boldsymbol{r}'}{|\boldsymbol{r}'|} = \frac{(-a\sin(t),b,a\cos(t)))}{(a^2+b^2)^{\frac{1}{2}}}$
$\mathbf{B} = \frac{\boldsymbol{r}'\times\boldsymbol{r}''}{|\boldsymbol{r}'\times\boldsymbol{r}''|} = \frac{(-b\sin(t),-a,b\cos(t)))}{(a^2+b^2)^{\frac{1}{2}}}$,
and $\mathbf{N} = \mathbf{B} \times \mathbf{T} = (-\cos(t),\, 0,\, -\sin(t))$. The arc length is $L = \int_0^{2\pi} |\boldsymbol{r}'|dt = 2\pi(a^2+b^2)^{\frac{1}{2}}$.

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3 If we use the series $\exp(s) = \sum_{k=0}^{\infty} s^k/k!$ with $s = t$ and $s = -t$, we find $\sinh(t) = \sum_{k=0}^{\infty} t^{2k+1}/(2k+1)!$, and $f(t) = \sum_{k=0}^{\infty} t^{2k}/(2k+1)!$, which converge respectively for s, and $t \in \Re$. Thus we have

$$I(x) = \int_0^x f(t)dt = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)(2k+1)!}, \quad x \in \Re.$$

Taking only terms up to x^5, we obtain

$$T_5(x) = x + \frac{x^3}{18} + \frac{x^5}{600}.$$

It follows that $I(\frac{1}{2}) \approx T_5(\frac{1}{2}) = 0.506997$.
