Math 264 Sample Midterm Test

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Instructions: *Please answer all 4 questions which carry equal marks. Explain your work clearly.*

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1. Consider the curve in $\Re^3$ given by $\boldsymbol{r}(t) = (a\cos(t), a\sin(t), bt)$, $0 \le t \le 4\pi$, where $a > 0$ and $b > 0$.

   (a) Give a rough sketch of this curve.

   (b) If $a = 2$ and $b = 3$, find the arc length of the curve for $t \in [0, 4\pi]$.

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2. Consider the curve given in polar coordinates by $r(\theta) = 1 - \cos(\theta), \theta \in [0, 2\pi]$.

   (a) Give a rough sketch of this curve.

   (b) Find the area inside the curve given in (a) and outside the curve $r = 1$.

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3. Consider the curve in $\Re^3$ given by $\boldsymbol{r}(t) = (t, t^2, t^3), t \ge 0$.

   (a) Give a rough sketch of this curve.

   (b) Find the plane that is orthogonal to the curve at $t = 1$.

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4. Consider the integral $I(x)$ given by $I(x) = \int\limits_0^x \sin(t^2)dt$, that is to say, $I(x) = \int \sin(x^2)dx$, and $I(0) = 0$.

   (a) Find a Taylor series for the integral $I(x)$

   (b) Now find a polynomial $T(x)$ to approximate $I(x)$ with error less than 10^{-7} for $0 \le x \le \frac{1}{2}$.

Solution Notes

1.

(a) The curve is a helix of radius a revolving about the z-axis. There are two complete turns and z rises by $4\pi b$ units as t advances from 0 to 4π.

(b) Arc length $L = \int_0^{4\pi} |\boldsymbol{r}'(t)| dt$. Meanwhile $|\boldsymbol{r}'(t)| = (a^2 + b^2)^{\frac{1}{2}}$. Thus we have $L = (a^2 + b^2)^{\frac{1}{2}} 4\pi = 4\sqrt{13}\pi$

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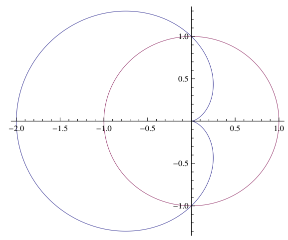
(a) The curve is a heart-shaped figure (cardoid) with axis the x-axis and tip at $(-2, 0)$.

(b) The cardoid intersects the circle $r = 1$ where $cos(\theta)$ vanishes, that is to



say at $\theta = \pi/2$ and $\theta = 3\pi/2$. Thus the required area is given by

$$A = \frac{1}{2} \int_{\pi/2}^{3\pi/2} \left[r^2(\theta) - 1 \right] d\theta = 2 + \pi/4.$$

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(a) The curve advances monotonically and (for $t > 1$) ‘faster’ in y than x and faster in z than y.

(b) The tangent line is along $\boldsymbol{r}'(1) = (1, 2, 3) = \mathbf{n}$ which is a normal to the plane. Meanwhile $\boldsymbol{r}(1) = (1, 1, 1) = \mathbf{p}$ is a point in the plane. Hence the equation of the plane is $(\mathbf{r} - \mathbf{p}) \cdot \mathbf{n} = 0$, that is to say
$(x - 1) + 2(y - 1) + 3(z - 1) = 0$.

4.

(a) Use the well-known (*) series for $\sin(s)$, set $s = t^2$, and integrate $\int_0^x (\ldots) dt$ term-by-term to obtain $I(x) = \sum_{n=0}^{\infty} \frac{x^{4n+3}}{(2n+1)!(4n+3)} (-1)^n$.

(b) For $0 \le x \le \frac{1}{2}$, $T(x) = x^3/3 - x^7/42 + x^{11}/1320$ with error (‘next term’) less than 10^{-7}.

(*) Mnemonic: the series for $\sin(s)$ is the odd part of the series for e^s, with alternating signs. Thus $\sin(s) = \sum_{n=0}^{\infty} \frac{s^{2n+1}}{(2n+1)!} (-1)^n$.
