Computational Applied Mathematics: Problems 2

Professor:	*Richard Hall*
Instructions:	*Please explain your solutions carefully.*
Due Date:	*5 March 2008*

(1.1) Write Maple code to generate (pseudo) random numbers Y on $[0, a]$ with density $\rho(y) = cy^2$, where c is a normalization constant. In particular, write a procedure `sqd(n, a)` that generates an array of n y-values. Use this array to plot an approximate graph of $\rho(y)$ on a mesh on $[0, a]$ of m intervals: the graph would join the 'histogram' heights at the m half points $\{(j - \frac{1}{2})h\}$, where $h = a/m$, and $j = 1, 2, \ldots, n$. Plot the graph for the values $a = 4, m = 10, n = 10000$.

(1.2) Derive the 3σ-rule and find the probable error, if X is a random variable with distribution $N(\mu, \sigma)$. That is to say, find (i) the probability that x lies within 3σ of μ, and (ii) the number b such that the probability that x is within $b\sigma$ of μ is $\frac{1}{2}$.

(1.3) Study the integral $\int_0^1 \cosh(x)dx$. This can, of course, be computed exactly; but we are going to try it by Monte-Carlo methods.

(a) First estimate the integral by the procedure `mu(n)` which uses uniform sampling. Find the results `m(10)` and `m(1000)` and the corresponding approximate σ and the 3σ-error.

(b) Now repeat the exercise using significance sampling and a procedure `mg(n)` . HINT: Consider the integral of $f(x) = \cosh(x) - 1$, and random sampling with the density $\rho(x) = 3x^2$ on $[0, 1]$. Why is this density suggested?

(1.4) Write a Maple procedure `binormal(n)` that uses the Box-Muller transformation to generate an array of n binormal pairs $[x, y]$, each approximately of type $N(0, 1)$. Plot a histogram of say X with 20 intervals on $[-4, 4]$ using $n = 10000$ points (similar to problem (1.1)). Compare this graph to a suitably normalized graph of the corresponding analytical density $\rho(x)$.

(1.5) A binormal density $N(0, 1) \times N(0, 1)$ is centred at the origin. A square of side $2a$ is placed in the plane with its centre a distance r from the origin and its side rotated to an angle θ to the x-axis. Suppose the probability mass in the square is given by $m(a, r, \theta)$. Use a Monte-Carlo method to study the graph of m with respect to θ, for fixed a and r. In particular, plot a graph of $m(1, 2, \theta)$ for $\theta \in [0, \pi/2]$. The analytic study of this curious problem of a square bucket placed under a Gaussian fountain may be found in the paper: [22] *Inequalites for the probability content of a rotated square and related convolutions*, Ann. Prob. **8**, 802-813 (1980), by Richard L. Hall, M. Kanter, and M. Perlman.