Department of Mathematics & Statistics

Math 366 & 601J Sec AA Final Exam December 2013

Professor: *Richard Hall*

Instructions: *Please answer all 5 questions which carry equal marks.*
Explain your work carefully.
Approved calculators are permitted.

1. Find all values of the expressions:
   (a) $\left(4\sqrt{3}+4i\right)^{\frac{1}{3}}$.
   (b) 5^i .

2. Consider the analytic function $f(z) = u(x,y) + iv(x,y)$,
   where $u(x,y) = x^2 - y^2 - x + 1$.
   (a) Find the harmonic conjugate $v(x,y)$ to the given $u(x,y)$.
   (b) Sketch the families of level curves $u(x,y) = c_1$ and $v(x,y) = c_2$.
   (c) Show that the two sets of curves in (b) are orthogonal.

3. Consider the function $f(z)$ defined by

$$f(z) = \begin{cases} \frac{e^z - 1}{z}, & z \neq 0 \\ 1, & z = 0. \end{cases}$$

   (a) Show that $f(z)$ may be redefined as an entire function by means of a suitable Taylor (Maclaurin) series.
   (b) Find $f'(z)$
   (c) What is the function $g(z) = \frac{1}{i\pi}\int_\gamma \frac{f(s)ds}{(s-z)^3}$, where γ is a simple closed positively oriented contour with z in its interior.
   (d) Find the first three terms of a power series in z for the function $1/f(z)$.

4. By means of a suitable contour integral, evaluate the real improper integral I given by

$$I = \int_0^\infty \frac{x^2 dx}{(x^2+9)(x^2+25)}.$$

5. Consider the family of integrals

$$J_n(a,b) = \int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^n}, \quad a > b > 0, \quad n = 1, 2, 3 \ldots$$

(a) Use contour integration to find an expression for $J_1(a,b)$.

(b) Use part (a) to find $J_2(a,b)$.

HINT: consider $\frac{\partial}{\partial a} J_1(a,b)$.

(c) Provide a recipe for $J_n(a,b)$.