Reprinted from JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS
Vol. 145, No. 2, January 15, 1990

Some Oscillating Integrands with Small Ripples

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Submitted by R. P. Boas

Received August 10, 1987

This article concerns the convergence of a class of real improper integrals of the form

$$I = \int_{\alpha}^{\infty} \frac{\sin(x)}{f(x)}\, dx, \tag{1}$$

where α is a constant, $f(x)$ is continuous and positive, and $\lim_{x \to \infty} f(x) = \infty$. If $f(x)$ increases sufficiently rapidly, then the integral is absolutely convergent. We shall not be concerned with this case. If $f(x)$ is *monotone*, then it is well known (see, for example, [2, p. 333]) that the integral I converges just like an alternating series of the form $\sum_{n=m}^{\infty} (-1)^n/f(n\pi)$. If the monotonicity condition of $f(x)$ is dropped, it is difficult to imagine how to retain sufficient control over $f(x)$ so that new convergence results would still have an interesting level of generality. This choice was resolved for us by a fundamental problem in mathematical physics in which the existence of discrete eigenvalues for the Dirac Hamiltonian operator hinges [1] on the convergence of integrals with the general form (1). The class of integrals we study in this article is illustrated by the following five examples, only *one* of which converges:

$$I_1 = \int_{5}^{\infty} \frac{\sin(x)}{\sqrt{x} + \sin(x)}\, dx \tag{2}$$

$$I_2 = \int_{5}^{\infty} \frac{\cos(x)}{\ln(x) + (\cos(x))/\sqrt{x}}\, dx \tag{3}$$

$$I_3 = \int_{5}^{\infty} \frac{\sin(x)}{\ln(x) + \cos(x)}\, dx \tag{4}$$

$$I_4 = \int_{5}^{\alpha} \frac{\sin(x)}{\ln(x) + \cos(x) + (\sin(x))/\sqrt{x}}\, dx \tag{5}$$

$$I_5 = \int_{5}^{\infty} \frac{\sin(x)}{\sqrt{x} + \cos(x) + (\sin(x))/\ln(x)}\, dx. \tag{6}$$

0022-247X/90 $3.00

In all these examples a relatively small ripple is added to a monotone unbounded function in the denominator. Sometimes such a small periodic disturbance is sufficient to upset the delicate balance between the positive and negative contributions to the integral. The theorems we shall prove will establish some convergence properties of integrals of this type.

In the first theorem we characterize the critical amplitude of an additive sinusoidal perturbation to the denominator of (1).

THEOREM 1. (i) $f, g \in C[0, \infty)$.

(ii) *f is monotone increasing, $f(x) > 0$, and $\lim_{x \to \infty} f(x) = \infty$.*

(iii) *g is positive and bounded, $0 < g(x) \leqslant M$.*

(iv) *$g(x)/f^2(x)$ is monotone decreasing.*

Then there exists a number $\alpha \geqslant 0$ such that the following two integrals either both converge or both diverge:

$$I_a = \int_\alpha^\infty \frac{\sin(x)}{f(x) + g(x)\sin(x)}\, dx, \qquad I_b = \int_b^\infty \frac{g(x)}{f^2(x)}\, dx.$$

By changes of variable respectively to $x' = x + \pi$ and $x' = x + \pi/2$, we see that the conclusion of Theorem 1 is not altered (i) if the "+" sign is changed to "−" in the denominator of the integrand of I_a or (ii) if *both* sine functions in the integral I_a are changed to cosine. If only *one* of the sine functions is changed to cosine, the outcome is quite different, as we shall see.

Before we prove Theorem 1 we shall need to prove the following lemma, which tells us that the convergence of the integral of a positive monotone function is not altered by the insertion into the integrand of a factor $\sin^2(x)$.

LEMMA 1. (i) $\phi \in C(0, \infty)$.

(ii) *$\phi(x) \geqslant 0$, and ϕ is monotone decreasing for $x \in [0, \infty)$.*

(iii) *α is a real constant, $\alpha \geqslant 0$.*

Then the following two integrals either both converge or both diverge:

$$\int_\alpha^\infty \sin^2(x)\, \phi(x)\, dx, \qquad \int_\alpha^\infty \phi(x)\, dx.$$

Proof. It is sufficient to suppose that $\alpha = 0$. We divide the half line into intervals of length π and employ the mean value theorem on the nth patch, $P_n = [n\pi, (n+1)\pi]$. By using the monotonicity of ϕ we find

$$(\pi/2)\, \phi((n+1)\pi) \leqslant \int_{P_n} \sin^2(x)\, \phi(x)\, dx \leqslant (\pi/2)\, \phi(n\pi)$$

and also

$$\pi\phi((n+1)\pi) \leqslant \int_{P_n} \phi(x)\,dx \leqslant \pi\phi(n\pi).$$

Hence the integrals in the lemma each have upper and lower estimates which differ, from one integral to the other, by a factor of two. Therefore, the integrals either both converge or both diverge. ∎

Now we turn to the proof of Theorem 1. Since f increases without bound and g is bounded, there exists a number $\alpha \geqslant 0$ such that the following inequalities are valid:

$$0 < f(x)/2 < f(x) + g(x)\sin(x) < 2f(x), \qquad x > \alpha. \tag{7}$$

We first assume that I_a converges. We may write I_a in the form

$$I_a = \int_\alpha^\infty \frac{\sin(x)}{f(x)}\,dx - \int_\alpha^\infty \frac{g(x)\sin^2(x)}{f(x)\{f(x)+g(x)\sin(x)\}}\,dx. \tag{8}$$

Now the first integral on the RHS of (8) is convergent because f increases monotonically without bound; therefore the convergence of I_a implies the convergence of the second integral. By using (7) we find that

$$\int_\alpha^\infty \frac{g(x)\sin^2(x)}{f(x)\{f(x)+g(x)\sin(x)\}}\,dx \geqslant \int_\alpha^\infty \frac{g(x)\sin^2(x)}{2f^2(x)}\,dx. \tag{9}$$

The function $g(x)/f^2(x)$ is monotone decreasing and consequently we may conclude from (9) and Lemma 1 that $\int_\alpha^\infty (g(x)/f^2(x))\,dx$ is convergent. This completes the proof of the first part of the theorem.

Now we *assume* that $\int_\alpha^\infty (g(x)/f^2(x))\,dx$ is convergent. By a similar argument to the first part of this proof, we can use (7) and Lemma 1 to show that the last integral on the RHS of (8) is convergent. Consequently, I_a is convergent. This completes the proof of Theorem 1.

It follows from Theorem 1 that the examples I_1 and I_2 above both diverge. The illustrations one can construct with this theorem are pedagogically interesting because they can involve, via the integral I_b of Theorem 1, various traditional examples of improper integrals whose positive integrands do not vanish quite fast enough for convergence. In the following, for example, we know from Theorem 1 that I_6 diverges because I_b does:

$$I_6 = \int_5^\infty \frac{\sin(x)}{\ln(\ln(x)) + (\sin(x))/x}\,dx, \qquad I_b = \int_5^\infty \frac{1}{x[\ln(\ln(x))]^2}\,dx. \tag{10}$$

However, the example I_3 poses a new problem because the main oscillation and the ripple in the integrand are out of phase. It turns out that the effect of this phase difference is sufficient to allow convergence. We shall prove the following

THEOREM 2. (i) $f \in C[0, \infty)$.

(ii) *f is monotone increasing, $f(x) > 0$, and $\lim_{x \to \infty} f(x) = \infty$.*

Then there exists a number $\alpha \geqslant 0$ such that the integral

$$\int_{\alpha}^{\infty} \frac{\sin(x)}{f(x) + \cos(x)}\, dx \text{ is convergent.}$$

This theorem immediately resolves the question concerning example I_3 above in favour of convergence. By changes of variable respectively to $x' = x + \pi$ and $x' = x + \pi/2$, we see that the conclusion of Theorem 2 is not altered (i) if the "+" sign is changed to "−" in the denominator of the integrand or (ii) if the sine and cosine functions in the integral are interchanged. In order to prove Theorem 2 we shall first establish a mean value lemma for the type of integral involved.

LEMMA 2. (i) *a and b are constants $a < b$.*

(ii) *$f, p, q \in C[a, b]$.*

(iii) *f is monotone increasing.*

(iv) *$p(x)$ is one-signed.*

(v) *$f(a) + q(x) > 0$.*

Then the integral

$$P = \int_a^b \frac{p(x)}{f(x) + q(x)}\, dx$$

has the representation

$$P = G(y) = \int_a^b \frac{p(x)}{f(y) + q(x)}\, dx, \qquad \text{for some } y \in [a, b].$$

Proof. It is sufficient to suppose that $p(x) > 0$. The function G is continuous. Consequently, $G(y)$ assumes all values between $G(a)$ and $G(b)$ as y varies between a and b. But the integral P lies between $G(a)$ and $G(b)$. Hence a number $y \in [a, b]$ exists such that $P = G(y)$. This establishes Lemma 2. ∎

Now we are able to prove Theorem 2. Since f is monotone increasing without bound, there exists a positive integer m such that $f(x) > 1$ for

$x \geqslant \alpha = m\pi$. We write the integral of the theorem as a sum of integrals K_n over the successive intervals $[n\pi, (n+1)\pi]$, $n = m, m+1, m+2, \ldots$. Thus, by applying Lemma 2 we find

$$K_n = \int_{n\pi}^{(n+1)\pi} \frac{\sin(x)}{f(x)+\cos(x)}\, dx = -\,[\ln|f(y_n)+\cos(x)|]_{n\pi}^{(n+1)\pi},$$

where $y_n \in [n\pi, (n+1)\pi]$. Hence K_n is given in general by the formula

$$K_n = (-1)^n F(y_n), \qquad \text{where} \quad F(y) = \ln|(f(y)+1)/(f(y)-1)|. \qquad (11)$$

Since f is monotone *increasing* without bound, F is therefore monotone *decreasing* to zero. Also the sequence $\{y_n\}_{n=m}^{\infty}$ is monotone increasing without bound. Consequently the sum $\sum_{n=m}^{\infty} K_n$ representing the integral in the theorem is a convergent alternating series. This establishes Theorem 2.

In our last result we show that the convergence of the integral of Theorem 2 can be upset by a small ripple. We shall prove

THEOREM 3. (i) $f, g \in C[0, \infty)$.

(ii) *f is monotone increasing, $f(x) > 0$, and $\lim_{x\to\infty} f(x) = \infty$.*

(iii) *$g(x)$ is positive and bounded, $0 < g(x) \leqslant M$.*

(iv) *$g(x)/f^2(x)$ is monotone decreasing.*

Then there exists a number $\alpha \geqslant 0$ such that the following two integrals either both converge or both diverge:

$$I_a = \int_{\alpha}^{\infty} \frac{\sin(x)}{f(x)+\cos(x)+g(x)\sin(x)}\, dx, \qquad I_b = \int_{\alpha}^{\infty} \frac{g(x)}{f^2(x)}\, dx.$$

Proof. This theorem and its proof are very similar to Theorem 1. We therefore only present an outline of the proof. Since f is positive and increases without bound, and g is bounded, there exists a number $\alpha \geqslant 0$ such that the functions $\{f(x)+\cos(x)\}$ and $\{f(x)+\cos(x)+g(x)\sin(x)\}$ are both positive for $x \geqslant \alpha$. If I_a converges, then we can rewrite it as the difference between the convergent integral which is the subject of Theorem 2 and the integral J given by

$$J = \int_{\alpha}^{\infty} \frac{g(x)\sin^2(x)}{\{f(x)+\cos(x)\}\{f(x)+\cos(x)+g(x)\sin(x)\}}\, dx.$$

The main task (for both sides of the proof) is to show that J converges if and only if the integral I_b converges. This is done, as in the proof of Theorem 1, by first estimating the denominator of the integrand of J, for large x, as an appropriate constant multiplied by the square of $f(x)$, and then making use of Lemma 1. ∎

Applications of Theorem 3 show that the integrals I_4 and I_5 both diverge whereas, for example, the integral

$$I_7 = \int_5^\infty \frac{\sin(x)}{\ln(x) + \cos(x) + (\sin(x))/x}\, dx \tag{12}$$

converges. It is easy to use the above theorems and lemmas to generate more examples and variations on this theme.

Corresponding results for conditionally convergent series are easily obtained by very similar arguments. The integral results which we have discussed could, of course, be generalized to accommodate functions that oscillate "like" sine and cosine, but are different from these functions. Similarly, we could also relax the continuity assumptions which are only necessitated by our elementary methods but not by the nature of the problem.

Acknowledgments

The author would like to thank the referee for constructive comments on this paper. Partial financial support of this work by the Natural Sciences and Engineering Research Council of Canada under Grant A3438 is also gratefully acknowledged.

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Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium