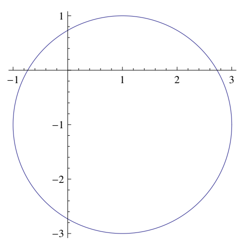
Math 366 Sec AA Midterm Test Solution Notes October 2013

Professor:	*Richard Hall*
Instructions:	*Please answer all 5 questions which carry equal marks.* *Duration: 1 hour. Please explain your work clearly.*

1. Let D be the region in C defined by $|z-1+i|<2$.
   (a) Sketch D.
   (b) Determine whether or not $f(z)=e^z/(z+1)$ is analytic in D.

(a) 

(b) $z_1=-1$ is the only singular point of $f(z)$ in C. But it is clear from (a) that z_1 is not in D. Hence $f(z)$ is analytic in D since e^z and $1/(z+1)$ are both analytic in D.

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2. Suppose that $f(z)=i\,v(x,y)$ is analytic and imaginary-valued in a domain D, and has the particular value $f(1+2i)=3i$. Find $f(z)$ and $f'(z)$.

Since $\Re(f(z))=u(x,y)=0$, the CR equations tell us that $v_x=v_y=0$. That is to say, $v(x,y)=c$, a constant. The given value of $f(1+2i)=3i$ implies $v=3$, that is to say, $f(z)=3i$, which in turn implies $f'(z)=0$.

3.
   (a) Find the distinct roots of $z^3=4\sqrt{2}(1-i)$.
   (b) Find the sum q of the distinct roots found in (a).

In polar form, $z^3=8e^{i\frac{\pi}{4}}$. Hence, $z=2e^{i(\frac{\pi}{12}+\frac{2k\pi}{3})}$, where $k=0,\pm1,\pm2,\ldots$ If $\omega=e^{i\frac{2\pi}{3}}$, then there are three distinct roots given by $\{z_1,\omega z_1,\omega^2 z_1\}$, where $z_1=2e^{i\frac{\pi}{12}}$. Thus $q=z_1(1+\omega+\omega^2)$. But $\omega^3=1$, so $\omega q=q$. Since $\omega\neq1$, therefore $q=0$.

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4. Define the function $\sin(z)$ in terms of z.
   (a) Express $\sin(z)$ in the form $\sin(z) = u(x,y) + i\,v(x,y)$.
   (b) Show that $u(x,y)$ and $v(x,y)$ of part (a) are harmonic functions.

$sin(z) \equiv \frac{1}{2i}(e^{iz} - e^{-iz}) = \sin(x)\cosh(y) + i\cos(x)\sinh(y) = u(x,y) + iv(x,y)$. Partial differentiation shows $u_{xx} + u_{yy} = 0 = v_{xx} + v_{yy}$.

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5. Consider the function $f(z) = z\,e^{z^2}$.
   (a) Assume that $\exp(z)$ is entire, and show that $f(z)$ is entire.
   (b) Let γ be the curve $z(t) = (1+i)t$, where $0 \le t \le 1$. Find $\int_\gamma f(z)dz$.

Since, polynomials and the exponential function are analytic everywhere, and since products and compositions of analytic functions are analytic, the given function $f(z)$ is entire. In particular, it has the (analytic) anti-derivative $F(z) = \frac{1}{2}e^{z^2}$. Thus line integrals are path-independent. In particular,

$$\int_0^{1+i} f(z)dz = F(z)|_0^{1+i} = F(1+i) - F(0) = \frac{1}{2}(e^{2i} - 1) = \frac{1}{2}(\cos(2) - 1 + i\sin(2)).$$

The same result can of course be obtained directly by an integration, for example, over the path $\gamma :\ z(t) = (1+i)t, \quad t \in [0,1]$.
