# Math 366 & 601J Sec AA  Midterm-Test Solution Notes November 2014

**Professor:** *Richard Hall*

**Instructions:** *Please answer all 5 questions which carry equal marks. Duration: 1 hour. Please explain your work clearly.*

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1. Consider the polynomial $p(z) = z^4 + 5z^2 - 50$, where $z \in C$. Now suppose that $z$ is restricted to the subset $B$ of $C$, where $|z| = 3$. Describe $B$ and show that for $z \in B$,

$$\left|\frac{1}{z^4 + 5z^2 - 50}\right| \leq \frac{1}{4}.$$

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$|p(z)| = |(z^2 - 5)(z^2 + 10)| = |z^2 - 5|\,|z^2 + 10| \geq ||z|^2 - 5|\,||z|^2 - 10| = 4 \times 1.$

Since $|z| = 3$ on $B$, we conclude $1/|p(z)| \leq 1/4.$

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2. Determine whether or not the following functions are analytic in suitable domains. When possible, find the derivative.

   (a) $f(z) = e^x e^{-iy}$

   (b) $g(z) = e^y e^{-ix}$ .

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(a) $f(z) = e^x(\cos y - i \sin y) = e^{\bar{z}}$ . But $\bar{z}$ is nowhere differentiable.

Thus $f(z)$ likewise is nowhere differentiable. [or use CR].

(b) $g(z) = u(x, y) + iv(x, y) = e^{-iz}$ . Since $\exp(z)$ is entire, $g'(z) = -ie^{-iz}$ , by the chain rule. [or use CR].

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3. Consider the function $u(x, y) = x^3 - 3xy^2 - x$.

   (a) Show that $u(x, y)$ is harmonic.

   (b) Find a harmonic conjugate $v(x, y)$ to $u(x, y)$ .

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(a) $u_{xx} + u_{yy} = 6x - 6x = 0.$

(b) By CR we have $v_x = -u_y = 6xy$. Thus $v = 3x^2y + k(y)$.

Differentiating and using CR we find $v_y = 3x^2 + k'(y) = u_x = -3x^2 - 3y^2 - 1$.

Therefore, $k'(y) = -3y^2 - 1 \Rightarrow k(y) = -y^3 - y + A$. Hence $v = 3x^2y - y^3 - y + A$, where $A \in R$ is an arbitrary constant. By transcendental meditation evidently $f(z) = u + iv = z^3 - z$.

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4.

(a) Express all values of $(\sqrt{3}+i)^{-i}$ in the form $a+ib$.

(b) Find $z$ if $\cosh(z)=\cos(4)$.

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(a) First note that $e^{i\pi/6}=\cos(\pi/6)+i\sin(\pi/6)=\frac{1}{2}(\sqrt{3}+i)$. Thus

$(\sqrt{3}+i)^{-i} = [2e^{i\pi/6}]^{-i} = e^{-i(\ln(2)+i\pi/6+i2n\pi} = e^{\pi/6+2n\pi}(\cos(\ln 2) - i\sin(\ln 2))$.

where, here and below, $n=\pm\{0,1,2,3,...\}$.

(b) $\cos(4)=\cosh(x+iy)=\cosh(x)\cosh(iy)+\sinh(x)\sinh(iy)=$

$\cosh(x)\cos(y)+i\sinh(x)\sin(y)$. There are two cases:

(i) $y=0 \Rightarrow \cos(4)=\cosh(x)\geq 1$. Not possible.

(ii) $x=0 \Rightarrow \cos(4)=\cosh(iy)=\cos(y)$. Hence $z=iy=i(\pm 4+2n\pi)$ .

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5. Consider the function $f(z)=z|z|^2$ .

(a) Does $f(z)$ have an antiderivative?

(b) Let $\gamma_1$ be the curve $z(t)=(1+2i)t$ , where $0\leq t\leq 1$ , oriented with increasing $t$. Find $\int_{\gamma_1} f(z)dz$ .

(c) Let $\gamma_2$ be the circular curve of radius 1 , centre the origin, and orientation anticlockwise. Find $\int_{\gamma_2} f(z)dz$ .

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(a) Since $F'(z)=f(z)$, both $F(z)$ and $f(z)$ must be analytic. But the given $f(z)$ involves the factor $|z|^2$ , which is nowhere differentiable. The answer is “no”.

(b) By a direct calculation, the first integral is

$\int((1+2i)t5t^2(1+2i)dt=-(15/4)+5i$.

(c) The curve $\gamma_2$ is given by $z(t)=e^{it}$. Consequently we find directly

$\int_{\gamma_2} g(z)dz=\int_0^{2\pi}(e^{it}\left|e^{it}\right|^2 e^{it}idt=\int_0^{2\pi}[-\sin(2t)+i\cos(2t)]=0$,

although $f(z)$ is not analytic.

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