6. Let $y(x) = v(x)/\sqrt{x}$. Then $y' = x^{-1/2}\,v' - x^{-3/2}\,v/2$ and $y'' = x^{-1/2}\,v'' - x^{-3/2}\,v' + 3\,x^{-5/2}\,v/4$. Substitution into the ODE results in

$$\left[x^{3/2}\,v'' - x^{1/2}\,v' + 3\,x^{-1/2}\,v/4\right] + \left[x^{1/2}\,v' - x^{-1/2}\,v/2\right] + (x^2 - \frac{1}{4})x^{-1/2}\,v = 0\,.$$

Simplifying, we find that

$$v'' + v = 0\,,$$

with general solution $v(x) = c_1 \cos x + c_2 \sin x$. Hence

$$y(x) = c_1 x^{-1/2} \cos x + c_2\, x^{-1/2} \sin x\,.$$

8. The absolute value of the ratio of consecutive terms is

$$\left|\frac{a_{2m+2}\,x^{2m+2}}{a_{2m}\,x^{2m}}\right| = \frac{|x|^{2m+2}\;2^{2m}(m+1)!\,m!}{|x|^{2m}\;2^{2m+2}(m+2)!(m+1)!} = \frac{|x|^2}{4(m+2)(m+1)}\,.$$

Applying the ratio test,

$$\lim_{m\to\infty}\left|\frac{a_{2m+2}\,x^{2m+2}}{a_{2m}\,x^{2m}}\right| = \lim_{m\to\infty}\frac{|x|^2}{4(m+2)(m+1)} = 0\,.$$

Hence the series for $J_1(x)$ converges absolutely for all values of x. Furthermore, since the series for $J_0(x)$ also converges absolutely for all x, term-by-term differentiation results in

$$J_0'(x) = \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m-1}}{2^{2m-1}\,m!(m-1)!} = \sum_{m=0}^{\infty}\frac{(-1)^{m+1}\,x^{2m+1}}{2^{2m+1}(m+1)!\,m!} =$$

$$= -\frac{x}{2}\sum_{m=0}^{\infty}\frac{(-1)^m\,x^{2m}}{2^{2m}(m+1)!\,m!}\,.$$

Therefore, $J_0'(x) = -J_1(x)$.

9.(a) Note that $x\,p(x) = 1$ and $x^2 q(x) = x^2 - \nu^2$, which are both analytic at $x = 0$. Thus $x = 0$ is a regular singular point. Furthermore, $p_0 = 1$ and $q_0 = -\nu^2$. Hence the indicial equation is $r^2 - \nu^2 = 0$, with roots $r_1 = \nu$ and $r_2 = -\nu$.

(b) Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=0}^{\infty}(r+n)a_n\,x^{r+n}$$

$$+\sum_{n=0}^{\infty}a_n x^{r+n+2} - \nu^2\sum_{n=0}^{\infty}a_n x^{r+n} = 0\,.$$

After adjusting the indices in the second-to-last series, we obtain

$$a_0\left[r(r-1) + r - \nu^2\right]x^r + a_1\left[(r+1)r + (r+1) - \nu^2\right]$$

$$+\sum_{n=2}^{\infty}\left[(r+n)(r+n-1)a_n + (r+n)a_n - \nu^2 a_n + a_{n-2}\right]x^{r+n} = 0.$$

Setting the coefficients equal to zero, we find that $a_1 = 0$, and

$$a_n = \frac{-1}{(r+n)^2 - \nu^2}\, a_{n-2}\,,$$

for $n \geq 2$. It follows that $a_3 = a_5 = \ldots = a_{2m+1} = \ldots = 0$. Furthermore, with $r = \nu$,

$$a_n = \frac{-1}{n(n+2\nu)}\, a_{n-2}\,.$$

So for $m = 1, 2, \ldots$,

$$a_{2m} = \frac{-1}{2m(2m+2\nu)}\, a_{2m-2} = \frac{(-1)^m}{2^{2m}\, m!(1+\nu)(2+\nu)\ldots(m-1+\nu)(m+\nu)}\, a_0\,.$$

Hence one solution is

$$y_1(x) = x^{\nu}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1+\nu)(2+\nu)\ldots(m-1+\nu)(m+\nu)} (\frac{x}{2})^{2m}\right].$$

(c) Assuming that $r_1 - r_2 = 2\nu$ is not an integer, simply setting $r = -\nu$ in the above results in a second linearly independent solution

$$y_2(x) = x^{-\nu}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!(1-\nu)(2-\nu)\ldots(m-1-\nu)(m-\nu)} (\frac{x}{2})^{2m}\right].$$