Math 370 Solution for Assignment 1 Q1

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Instructions: *Please explain your solutions carefully.*

1.1a(i) Suppose $f(x)$ is (i) continuously differentiable for all $x \in \Re$, and (ii) the function f satisfies Cauchy's functional equation $f(x+y) = f(x) + f(y)$. Prove that $f(x) = f(1)x$.

Solution: $f(x+h) - f(x) = f(h)$. Thus $f'(x) = f'(0)$. Hence $f(x) = f'(0)x + c$. Since $f(0+0) = f(0) + f(0)$, therefore $f(0) = 0$, and $c = 0$. Meanwhile, $f(1) = f'(0)$. Thus $f(x) = f(1)x$.

1.1a(ii) Now solve Cauchy's equation under the weaker assumption that $f(x)$ is continuous. HINT: first solve for fractions, then recall that $x_n \to x \Rightarrow f(x_n) \to f(x)$.

Let m and n be positive integers. Since $f(n) = f(1+n-1) = f(1) + f(n-1)$, we see that $f(n) = nf(1)$. By similar reasoning we find $f(n/m) = (n/m)f(1)$. Every real number can be expressed as the limit of a sequence $\{x_n\}$ of rational numbers, and we know that $f(x_n) = x_n f(1) \to xf(1)$. Meanwhile, the continuity of $f(x)$ implies $\lim f(x_n) \to f(x)$. Consequently, $f(x) = xf(1)$ for every real number x.