Math 370 Assignment 5

Sec. 3.6

17. Note that $g(x) = \ln x$. The functions $y_1(x) = x^2$ and $y_2(x) = x^2 \ln x$ are solutions of the homogeneous equation, as verified by substitution. The Wronskian of the solutions is $W(y_1, y_2) = x^3$. Using the method of variation of parameters, the particular solution is $Y(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$, in which

$$u_1(x) = -\int \frac{x^2 \ln x (\ln x)}{W(x)} dx = -(\ln x)^3/3$$

$$u_2(x) = \int \frac{x^2(\ln x)}{W(x)} dx = (\ln x)^2/2.$$

Therefore $Y(x) = -x^2(\ln x)^3/3 + x^2(\ln x)^3/2 = x^2(\ln x)^3/6$.

Problem (30) [Solution with help from (28), quoted first].

28. Let $y(t) = v(t)y_1(t)$, in which $y_1(t)$ is a solution of the homogeneous equation. Substitution into the given ODE results in

$$v''y_1 + 2v'y_1' + vy_1'' + p(t)\,[v'y_1 + vy_1'] + q(t)vy_1 = g(t)\,.$$

By assumption, $y_1'' + p(t)y_1 + q(t)y_1 = 0$, hence $v(t)$ must be a solution of the ODE

$$v''y_1 + [2y_1' + p(t)y_1]\, v' = g(t)\,.$$

Setting $w = v'$, we also have $w'y_1 + [2y_1' + p(t)y_1]\, w = g(t)$.

30. First write the equation as $y'' + 7t^{-1}y + 5t^{-2}\, y = t^{-1}$. As shown in Problem 28, the function $y(t) = t^{-1}v(t)$ is a solution of the given ODE as long as v is a solution of

$$t^{-1}\, v'' + \left[-2t^{-2} + 7t^{-2}\right] v' = t^{-1}\,,$$

that is, $v'' + 5t^{-1}\, v' = 1$. This ODE is linear and first order in v'. The integrating factor is $\mu = t^5$. The solution is $v' = t/6 + c\,t^{-5}$. Direct integration now results in $v(t) = t^2/12 + c_1 t^{-4} + c_2$. Hence $y(t) = t/12 + c_1 t^{-5} + c_2 t^{-1}$.

Sec. 3.7

15. The general solution of the system is $u(t) = A \cos \gamma(t - t_0) + B \sin \gamma(t - t_0)$. Invoking the initial conditions, we have $u(t) = u_0 \cos \gamma(t - t_0) + (u_0'/\gamma) \sin \gamma(t - t_0)$. Clearly, the functions $v = u_0 \cos \gamma(t - t_0)$ and $w = (u_0'/\gamma) \sin \gamma(t - t_0)$ satisfy the given criteria.

Sec 3.8

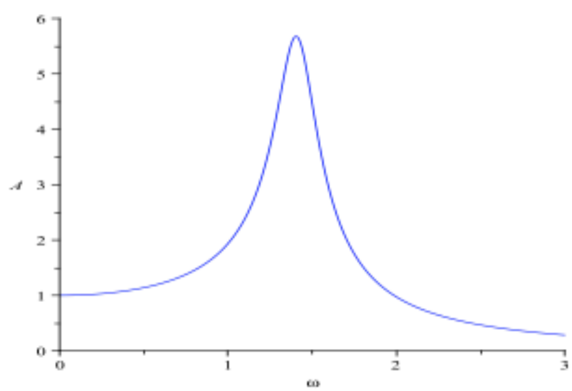
17.(a) The steady state part of the solution $U(t) = A\cos\omega t + B\sin\omega t$ may be found by substituting this expression into the differential equation and solving for A and B. We find that

$$A = \frac{32(2-\omega^2)}{64 - 63\omega^2 + 16\omega^4}, \qquad B = \frac{8\omega}{64 - 63\omega^2 + 16\omega^4}.$$

(b) The amplitude is

$$A = \frac{8}{\sqrt{64 - 63\omega^2 + 16\,\omega^4}}.$$

(c)



(d) See Problem 13. The amplitude is maximum when the denominator of A is minimum. That is, when $\omega = \omega_{max} = 3\sqrt{14}/8 \approx 1.4031$. Hence $A = 64/\sqrt{127}$.

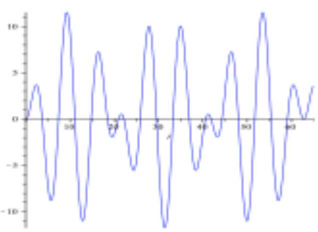
18.(a) The homogeneous solution is $u_c(t) = A\cos\, t + B\sin\, t$. Based on the method of undetermined coefficients, the particular solution is

$$U(t) = \frac{3}{1-\omega^2}\,\cos\,\omega t\,.$$

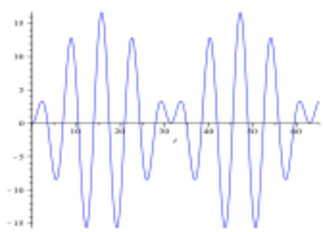
Hence the general solution of the ODE is $u(t) = u_c(t) + U(t)$. Invoking the initial conditions, we find that $A = 3/(\omega^2 - 1)$ and $B = 0$. Hence the response is

$$u(t) = \frac{3}{1-\omega^2}\,[\cos\,\omega t - \cos\, t\,]\,.$$

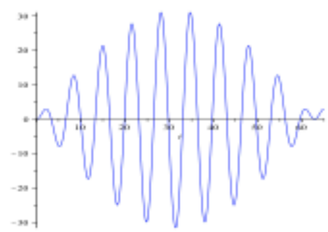
(b)



(a) $\omega = 0.7$



(b) $\omega = 0.8$



(c) $\omega = 0.9$

Note that

$$u(t) = \frac{6}{1-\omega^2}\,\sin\left[\frac{(1-\omega)t}{2}\right]\sin\left[\frac{(\omega+1)t}{2}\right].$$