Math 370 Assignment 7
Section 5.1

7. Apply the ratio test :

$$\lim_{n\to\infty} \frac{\left|3^n(n+1)^2(x+2)^{n+1}\right|}{\left|3^{n+1}n^2(x+2)^n\right|} = \lim_{n\to\infty} \frac{(n+1)^2}{3n^2}\left|(x+2)\right| = \frac{1}{3}\left|(x+2)\right|.$$

Hence the series converges absolutely for $\frac{1}{3}|x+2| < 1$, or $|x+2| < 3$. The radius of convergence is $\rho = 3$. At $x = -5$ and $x = +1$, the series diverges, since the n-th term does not approach zero.

16. We have $f(x) = 1/(1-x)$, $f'(x) = 1/(1-x)^2$, $f''(x) = 2/(1-x)^3, \ldots$ with $f^{(n)}(x) = n!/(1-x)^{n+1}$, for $n \geq 1$. It follows that $f^{(n)}(2) = (-1)^{n+1}n!$ for $n \geq 0$. Hence the Taylor expansion about $x_0 = 2$ is

$$\frac{1}{1-x} = -\sum_{n=0}^{\infty}(-1)^n(x-2)^n.$$

Applying the ratio test,

$$\lim_{n\to\infty} \frac{\left|(x-2)^{n+1}\right|}{\left|(x-2)^n\right|} = \lim_{n\to\infty}|x-2| = |x-2|.$$

The series converges absolutely for $|x-2| < 1$, but diverges at $x = 1$ and $x = 3$.

20. Shifting the index in the second series, that is, setting $n = k+1$,

$$\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{n=1}^{\infty} a_{n-1}x^n.$$

Hence

$$\begin{aligned}\sum_{k=0}^{\infty} a_{k+1}x^k + \sum_{k=0}^{\infty} a_k x^{k+1} &= \sum_{k=0}^{\infty} a_{k+1}x^k + \sum_{k=1}^{\infty} a_{k-1}x^k \\ &= a_1 + \sum_{k=1}^{\infty}(a_{k+1} + a_{k-1})x^{k+1}.\end{aligned}$$

24. Clearly,

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^n .$$

Shifting the index in the first series, that is, setting $k = n-2$,

$$\begin{aligned}\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} &= \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}\,x^k \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n .\end{aligned}$$

Hence

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}\,x^n - \sum_{n=2}^{\infty} n(n-1)a_n\,x^n .$$

Note that when $n=0$ and $n=1$, the coefficients in the second series are zero. So

$$(1-x^2)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - n(n-1)a_n\right]x^n .$$

Section 5.2

1.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n\right] x^n = 0\,.$$

Equating all the coefficients to zero,

$$(n+2)(n+1)a_{n+2} - a_n = 0\,, \qquad n = 0, 1, 2, \ldots\,.$$

We obtain the recurrence relation

$$a_{n+2} = \frac{a_n}{(n+1)(n+2)}\,, \qquad n = 0, 1, 2, \ldots\,.$$

The subscripts differ by two, so for $k = 1, 2, \ldots$

$$a_{2k} = \frac{a_{2k-2}}{(2k-1)2k} = \frac{a_{2k-4}}{(2k-3)(2k-2)(2k-1)2k} = \ldots = \frac{a_0}{(2k)!}$$

and

$$a_{2k+1} = \frac{a_{2k-1}}{2k(2k+1)} = \frac{a_{2k-3}}{(2k-2)(2k-1)2k(2k+1)} = \ldots = \frac{a_1}{(2k+1)!}\,.$$

Hence

$$y = a_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} + a_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}\,.$$

The linearly independent solutions are

$$y_1 = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots = \cosh x$$

$$y_2 = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots = \sinh x\,.$$

(c) The Wronskian at 0 is 1.