Math 370 Asssignment 8

First jump to Sec. 5.4

5. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r(r-1) - r + 1 = r^2 - 2r + 1$. The root is $r = 1$, with multiplicity two. Hence the general solution, for $x \neq 0$, is $y = (c_1 + c_2 \ln|x|)\, x$.

6. Substitution of $y = (x-1)^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r^2 + 7r + 12$. The roots are $r = -3, -4$. Hence the general solution, for $x \neq 1$, is $y = c_1 (x-1)^{-3} + c_2 (x-1)^{-4}$.

8. Substitution of $y = x^r$ results in the quadratic equation $F(r) = 0$, where $F(r) = r^2 - 3r + 3$. The roots are complex, with $r = (3 \pm i\sqrt{3})/2$. Hence the general solution, for $x \neq 0$, is

$$y = c_1 |x|^{3/2} \cos(\frac{\sqrt{3}}{2} \ln|x|) + c_2 |x|^{3/2} \sin(\frac{\sqrt{3}}{2} \ln|x|).$$

Now return to Sec. 5.2

6.(a,b) Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n .$$

Substitution into the ODE results in

$$(2+x^2) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n - x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 4 \sum_{n=0}^{\infty} a_n x^n = 0 .$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n .$$

It follows that

$$4a_0 + 4a_2 + (3a_1 + 12a_3)x +$$

$$+ \sum_{n=2}^{\infty} \left[2(n+2)(n+1) a_{n+2} + n(n-1) a_n - n\, a_n + 4a_n \right] x^n = 0 .$$

Equating the coefficients to zero, we find that $a_2 = -a_0$, $a_3 = -a_1/4$, and

$$a_{n+2} = -\frac{n^2 - 2n + 4}{2(n+2)(n+1)}\, a_n , \quad n = 0, 1, 2, \ldots .$$

The indices differ by two, so for $k = 0, 1, 2, \ldots$

$$a_{2k+2} = -\frac{(2k)^2 - 4k + 4}{2(2k+2)(2k+1)}\, a_{2k}$$

and

$$a_{2k+3} = -\frac{(2k+1)^2 - 4k + 2}{2(2k+3)(2k+2)}\, a_{2k+1} .$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - x^2 + \frac{x^4}{6} - \frac{x^6}{30} + \ldots$$

$$y_2(x) = x - \frac{x^3}{4} + \frac{7x^5}{160} - \frac{19x^7}{1920} + \ldots .$$

(c) The Wronskian at 0 is 1.

9.(a,b,d) Let $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$. Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

and

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n.$$

Substitution into the ODE results in

$$(1+x^2) \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n - 4x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 6 \sum_{n=0}^{\infty} a_n x^n = 0\,.$$

Before proceeding, write

$$x^2 \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2}\, x^n = \sum_{n=2}^{\infty} n(n-1) a_n x^n$$

and

$$x \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=1}^{\infty} n\, a_n x^n.$$

It follows that

$$6a_0 + 2a_2 + (2a_1 + 6a_3)x +$$

$$+ \sum_{n=2}^{\infty} \left[(n+2)(n+1) a_{n+2} + n(n-1) a_n - 4n\, a_n + 6 a_n\right] x^n = 0\,.$$

Setting the coefficients equal to zero, we obtain $a_2 = -3a_0$, $a_3 = -a_1/3$, and

$$a_{n+2} = -\frac{(n-2)(n-3)}{(n+1)(n+2)}\, a_n\,, \quad n = 0, 1, 2, \ldots\,.$$

Observe that for $n = 2$ and $n = 3$, we obtain $a_4 = a_5 = 0$. Since the indices differ by two, we also have $a_n = 0$ for $n \geq 4$. Therefore the general solution is a polynomial

$$y = a_0 + a_1 x - 3a_0 x^2 - a_1 x^3/3\,.$$

Hence the linearly independent solutions are

$$y_1(x) = 1 - 3x^2 \quad \text{and} \quad y_2(x) = x - x^3/3\,.$$

(c) The Wronskian is $(x^2+1)^2$. At $x = 0$ it is 1.