Math 370 Assignment 9

Sec. 5.4

38. Substitution of $y = x^r$ results in the quadratic equation $r^2 + (\alpha - 1)r + 5/2 = 0$. Formally, the roots are given by

$$r = \frac{1 - \alpha \pm \sqrt{\alpha^2 - 2\alpha - 9}}{2} = \frac{1 - \alpha \pm \sqrt{(\alpha - 1 - \sqrt{10})(\alpha - 1 + \sqrt{10})}}{2}.$$

(i) The roots will be complex if $|1 - \alpha| < \sqrt{10}$. For solutions to approach zero, as $x \to \infty$, we need $-\sqrt{10} < 1 - \alpha < 0$.

(ii) The roots will be equal if $|1 - \alpha| = \sqrt{10}$. In this case, all solutions approach zero as long as $1 - \alpha = -\sqrt{10}$.

(iii) The roots will be real and distinct if $|1 - \alpha| > \sqrt{10}$. It follows that

$$r_{max} = \frac{1 - \alpha + \sqrt{\alpha^2 - 2\alpha - 9}}{2}.$$

For solutions to approach zero, we need $1 - \alpha + \sqrt{\alpha^2 - 2\alpha - 9} < 0$. That is, $1 - \alpha < -\sqrt{10}$. Hence all solutions approach zero, as $x \to \infty$, as long as $\alpha > 1$.

Sec. 5.5

1.(a) $P(x) = 0$ when $x = 0$. Since the three coefficients have no common factors, $x = 0$ is a singular point. Near $x = 0$,

$$\lim_{x\to 0} x\,p(x) = \lim_{x\to 0} x\frac{1}{2x} = \frac{1}{2}\,.$$

$$\lim_{x\to 0} x^2 q(x) = \lim_{x\to 0} x^2\frac{1}{2} = 0\,.$$

Hence $x = 0$ is a regular singular point.

(b) Let

$$y = x^r(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots) = \sum_{n=0}^{\infty} a_n\, x^{r+n}.$$

Then

$$y' = \sum_{n=0}^{\infty}(r+n)a_nx^{r+n-1}$$

and

$$y'' = \sum(r+n)(r+n-1)a_nx^{r+n-2}.$$

Substitution into the ODE results in

$$2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n-1}+\sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1}+\sum_{n=0}^{\infty}a_n x^{r+n+1}=0\,.$$

That is,

$$2\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n}+\sum_{n=0}^{\infty}(r+n)a_n x^{r+n}+\sum_{n=2}^{\infty}a_{n-2}\,x^{r+n}=0\,.$$

It follows that

$$a_0\,[2r(r-1)+r]\,x^r+a_1\,[2(r+1)r+r+1]\,x^{r+1}$$
$$+\sum_{n=2}^{\infty}\left[2(r+n)(r+n-1)a_n+(r+n)a_n+a_{n-2}\right]x^{r+n}=0\,.$$

Assuming that $a_0\neq 0$, we obtain the indicial equation $2r^2-r=0$, with roots $r_1=1/2$ and $r_2=0$. It immediately follows that $a_1=0$. Setting the remaining coefficients equal to zero, we have

$$a_n=\frac{-a_{n-2}}{(r+n)\,[2(r+n)-1]}\,,\ \ n=2,3,\ldots\,.$$

(c) For $r=1/2$, the recurrence relation becomes

$$a_n=\frac{-a_{n-2}}{n(1+2n)}\,,\quad n=2,3,\ldots\,.$$

Since $a_1=0$, the odd coefficients are zero. Furthermore, for $k=1,2,\ldots$,

$$a_{2k}=\frac{-a_{2k-2}}{2k(1+4k)}=\frac{a_{2k-4}}{(2k-2)(2k)(4k-3)(4k+1)}=\frac{(-1)^k a_0}{2^k\,k!\,5\cdot 9\cdot 13\ldots(4k+1)}\,.$$

(d) For $r=0$, the recurrence relation becomes

$$a_n=\frac{-a_{n-2}}{n(2n-1)}\,,\quad n=2,3,\ldots\,.$$

Since $a_1=0$, the odd coefficients are zero, and for $k=1,2,\ldots$,

$$a_{2k}=\frac{-a_{2k-2}}{2k(4k-1)}=\frac{a_{2k-4}}{(2k-2)(2k)(4k-5)(4k-1)}=\frac{(-1)^k a_0}{2^k\,k!\,3\cdot 7\cdot 11\ldots(4k-1)}\,.$$

The two linearly independent solutions are

$$y_1(x)=\sqrt{x}\left[1+\sum_{k=1}^{\infty}\frac{(-1)^k\,x^{2k}}{2^k\,k!\,5\cdot 9\cdot 13\ldots(4k+1)}\right]$$

$$y_2(x)=1+\sum_{k=1}^{\infty}\frac{(-1)^k\,x^{2k}}{2^k\,k!\,3\cdot 7\cdot 11\ldots(4k-1)}\,.$$

3.(a) Note that $x\,p(x) = 0$ and $x^2 q(x) = x$, which are both analytic at $x = 0$.

(b) Set $y = x^r(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots)$. Substitution into the ODE results in

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0\,,$$

and after multiplying both sides of the equation by x,

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n\,x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n} = 0\,.$$

It follows that

$$a_0\,[r(r-1)]\,x^r + \sum_{n=1}^{\infty}\left[(r+n)(r+n-1)a_n + a_{n-1}\right]x^{r+n} = 0\,.$$

Setting the coefficients equal to zero, the indicial equation is $r(r-1) = 0$. The roots are $r_1 = 1$ and $r_2 = 0$. Here $r_1 - r_2 = 1$. The recurrence relation is

$$a_n = \frac{-a_{n-1}}{(r+n)(r+n-1)}\,, \quad n = 1, 2, \ldots\,.$$

(c) For $r = 1$,

$$a_n = \frac{-a_{n-1}}{n(n+1)}\,, \quad n = 1, 2, \ldots\,.$$

Hence for $n \geq 1$,

$$a_n = \frac{-a_{n-1}}{n(n+1)} = \frac{a_{n-2}}{(n-1)n^2(n+1)} = \ldots = \frac{(-1)^n a_0}{n!(n+1)!}\,.$$

Therefore one solution is

$$y_1(x) = x\sum_{n=0}^{\infty}\frac{(-1)^n\,x^n}{n!(n+1)!}\,.$$