Mast 330 / Math 370 Sec A Final Exam December 2004

Professor: *Richard Hall*
Instructions: *Please answer all 5 questions. Explain your working carefully. Calculators are permitted. [Lined booklets]*

1.
   (a) Find the general solution to the following differential equation and also a particular solution satisfying the given initial condition:

$$xy' - y = xy^2, \quad y(1) = 2.$$

   (b) If $y' = x + y^2$, and $y(0) = 1$, find an approximation for $y(0.2)$.

2. A certain population obeys the growth equation $y' = ky(1 - \alpha y)$, where k and α are positive parameters. Suppose that $y(0) = 100$, $\lim_{t\to\infty} y(t) = 1000$, and $y(10) = 200$. Find the values of α and k. [HINT: First find α; then solve the differential equation with k not yet determined; now use the solution and $y(10)$ to find k.]

3.
   (a) Find the orthogonal trajectories for the family of exponential curves given by $y = ce^x$, where $c \in \Re$. Sketch the two families.
   (b) Find the general solution to the differential equation:

$$(2x^3 + 2xy^2 + 2x)dx + 2ydy = 0.$$

4. Consider the forced oscillations of a damped spring-mass system represented by the differential equation

$$y'' + 2by' + y = \cos(\omega t),$$

where $y(t)$ is the position of the mass at time t, $b \geq 0$ is a friction parameter, and ω is a positive input-frequency parameter.

(a) Solve this problem in the frictionless case $b = 0$, with the initial conditions $y(0) = 0$, $y'(0) = 0$, and $\omega \neq 1$. In particular, find the solution in the limit $\omega \to 1$. [HINT: you can either find the limit of the solution for $\omega \neq 1$, or simply re-solve for the case $\omega = 1$.]

(b) Describe the general solution of this problem in the case $0 < b < 1$. In this case, find the steady-state solution approached in the limit $t \to \infty$.

5. By using Laplace transforms or otherwise solve the following initial-value problem in which a linear system, initially at rest, is stimulated by a delayed vibration $g(t)$:

$$y'''(t) - y''(t) = g(t), \quad t \geq 0,$$

where

$$y(0) = y'(0) = y''(0) = 0,$$

and

$$g(t) = \begin{cases} 0 \text{ for } t < 2, \\ 5\sin(t-2) \text{ for } t \geq 2. \end{cases}$$

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$
1	$\frac{1}{s},\ s > 0$
e^{at}	$\frac{1}{s-a},\ s > a$
$t^n, \quad n$ integer > 0	$\frac{n!}{s^{n+1}},\ s > 0$
$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}},\ s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2},\ s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2},\ s > 0$
$\sinh(bt)$	$\frac{b}{s^2-b^2},\ s > \|b\|$
$\cosh(bt)$	$\frac{s}{s^2-b^2},\ s > \|b\|$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2},\ s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2},\ s > a$
$e^{at}f(t)$	$F(s-a),\ s > a$
$(-t)^n f(t), \quad n$ integer > 0	$F^{(n)}(s)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \ldots - f^{(n-1)}(0)$
$f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right),\ c > 0$
$u_c(t)$	$\frac{e^{-cs}}{s},\ c \geq 0$
$u_c(t) f(t-c)$	$e^{-cs} F(s),\ c \geq 0$
$\delta(t-c)$	$e^{-cs},\ c \geq 0$
$\int_0^t f(t-\tau) g(\tau) d\tau$	$F(s)G(s)$