Mast 330 / Math 370 Sec A   Final Exam   December 2005

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Instructions: *Please answer all 5 questions. Explain your working carefully. Calculators are permitted. [Lined booklets]*

1. Consider the differential equation

$$y' = \frac{y}{x+y}.$$

   (a) Find the general solution, and verify that it satisfies the differential equation. [Note: your solution may be in implicit form].

   (b) Find a particular solution satisfying the initial condition $y(0) = 2.$:

2. Consider the differential equation

$$y' = (1+y^2)x^2.$$

   (a) Show that if y_1 is a solution, then $2y_1$ is *not* a solution.

   (b) If $y(1) = 1$, find $y'(1)$ and $y''(1)$ and hence, without solving the differential equation, find a polynomial of degree 2 in $(x-1)$ to approximate $y(x)$ near $x = 1$.

   (c) Find the general solution y.

3.

   (a) Find the general solution to the following differential equation

$$(y^2e^x + 2yx)dx + (3ye^x + 2x^2)dy = 0.$$

   (b) Find the orthogonal trajectories for the family of ellipses given by

$$\frac{x^2}{4} + \frac{y^2}{9} = a^2.$$

   Sketch the two families roughly on one diagram.

4. Consider the differential equation

$$y'' - 4y' + 4y = \frac{e^{2x}}{1+x^2}.$$

   (a) Find the general solution.

   (b) Find the particular solutions that vanish at $x = 0$.

5. Consider the following initial-value problem which represents a spring-mass system responding to a sequence of two impulses:

$$y'' + y = 2\delta_1(t) + 3\delta_4(t),$$

   (a) Find the general solution.

   (b) Find a particular solution which is initially at rest, that is to say, satisfying $y(0) = y'(0) = 0$. In this case, find the values $y(2)$ and $y(5)$.

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$
1	$\frac{1}{s},\ s > 0$
e^{at}	$\frac{1}{s-a},\ s > a$
$t^n, \quad n$ integer > 0	$\frac{n!}{s^{n+1}},\ s > 0$
$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}},\ s > 0$
$\sin(bt)$	$\frac{b}{s^2+b^2},\ s > 0$
$\cos(bt)$	$\frac{s}{s^2+b^2},\ s > 0$
$\sinh(bt)$	$\frac{b}{s^2-b^2},\ s > \|b\|$
$\cosh(bt)$	$\frac{s}{s^2-b^2},\ s > \|b\|$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2},\ s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2},\ s > a$
$e^{at}f(t)$	$F(s-a),\ s > a$
$(-t)^n f(t), \quad n$ integer > 0	$F^{(n)}(s)$
$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - \ldots - f^{(n-1)}(0)$
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right),\ c > 0$
$u_c(t)$	$\frac{e^{-cs}}{s},\ c \geq 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s),\ c \geq 0$
$\delta(t-c)$	$e^{-cs},\ c \geq 0$
$\int_0^t f(t-\tau)g(\tau)d\tau$	$F(s)G(s)$