Mast 330 /Math 370 Midterm Test 27 October 2004

Professor: *Richard Hall*
Instructions: *Please answer all 4 questions.*
Explain your work clearly. Calculators are permitted.

1. Solve the initial-value problem $y' = e^{-y} \cosh(x)$, $x \geq 0$, $y(0) = 1$, and find the value of $y(5)$. [Recall: $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$].

Solution [1]: *The equation is separable. We have* $\int e^y dy = \int \cosh(x) dx$, *that is to say* $e^y = \sinh(x) + C$, *or* $y = \ln(\sinh(x) + C)$. *The IC imply* $C = e$. *Thus* $y(5) = \ln(\sinh(5) + e) \approx 4.34279$.

2. Find the general solution to the following differential equation and also a particular solution satisfying $y(\pi) = 1$:

$$xy' + 2y = \frac{\cos(x)}{x}, \quad x > 0.$$

Solution [2]: *The equation may be written in the general linear form* $y' + py = g$. *Thus*

$$y' + \frac{2}{x}y = \frac{\cos(x)}{x^2}.$$

The integrating factor is then $\mu = exp(\int pdx) = x^2$. *The general solution is provided by the formula* $y = \left(\int \mu g dx + C\right)/\mu = (\sin(x) + C)/x^2$. *The IC imply* $y(\pi) = C/\pi^2 = 1$. *Thus we have* $C = \pi^2$.

3. Consider the differential equation

$$2xdx + (2y + x^2 + y^2)dy = 0.$$

(a) Show that the equation is *not* exact as it stands but can me made exact by use of a suitable integrating factor μ. Find μ.

(b) Find the general solution of the equation, and also a particular solution satisfying $y(0) = 2$.

Solution [3]: *We write the given equation in the form* $Mdx + Ndy = 0$. *We note that* $Q = M_y - N_x = -2x \neq 0$ *implies that the equation is not exact. We see that* Q/N *is not a function of* x *, but* $-Q/M$ *is a functon of* y *. Hence an integrating factor* $\mu(y)$ *exists and we have* $\mu'/\mu = -Q/M = 1$. *Thus we find* $\mu = e^y$ *is a suitable integrating factor. The new exact differential form is*

$$dF(x,y) = F_x dx + F_y dy = e^y(2x)dx + e^y(2y + x^2 + y^2)dy = 0.$$

Since integrating F_y *immediately requires integrating by parts, we instead integrate* F_x *and find*

$$F(x,y) = \int 2xe^y \partial x = x^2 e^y + k(y),$$

where $k(y)$ *is an unknown function. If we now differentiate this expression for* F *we find*

$$(2y + y^2 + x^2)e^y = F_y = x^2 e^y + k'(y).$$

This implies $k(y) = \int e^y(2y+y^2)dy = y^2 e^y$ *[after a necessary integration by parts]. Finally we conclude* $(x^2 + y^2)e^y = C$, *and the IC imply* $C = 4e^2$.

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4. Consider the following differential equation which describes the vibrations of a spring-mass system:

$$4y''(t) + 4y'(t) + y(t) = 0, \quad t \geq 0.$$

   (a) Find the general solution.

   (b) Find a particular solution satisfying the initial conditions $y(0) = 1$, $y'(0) = -2$ and provide a rough sketch of the graph of $y(t)$ for $t \in [0, 20]$. When is $y = 0$?

Solution [4]: *This equation is a second order homogeneous linear equation with constant coefficients. Trying* $y = e^{rt}$ *as a solution implies* $4r^2 + 4r + 1 = 0$. *This means that* $r = -\frac{1}{2}$, *twice. The solution obtained is* $y_1 = e^{-\frac{1}{2}t}$. *Since the root is repeated, we obtain another linearly independent solution in the form* $y_2 = ty_1$. *Thus the general solution is given by* $y = e^{-\frac{1}{2}t}(A + Bt)$, *where* A *and* B *are arbitrary constants. The IC imply* $A = 1$ *and* $B = -3/2$. *Thus the particular solution sought is* $y = e^{-\frac{1}{2}t}(1 - 3t/2)$. *This function vanishes when* $t = 2/3$ *(and also in the limit* $t \to \infty$ *). The graph is shown in Fig.(1)*

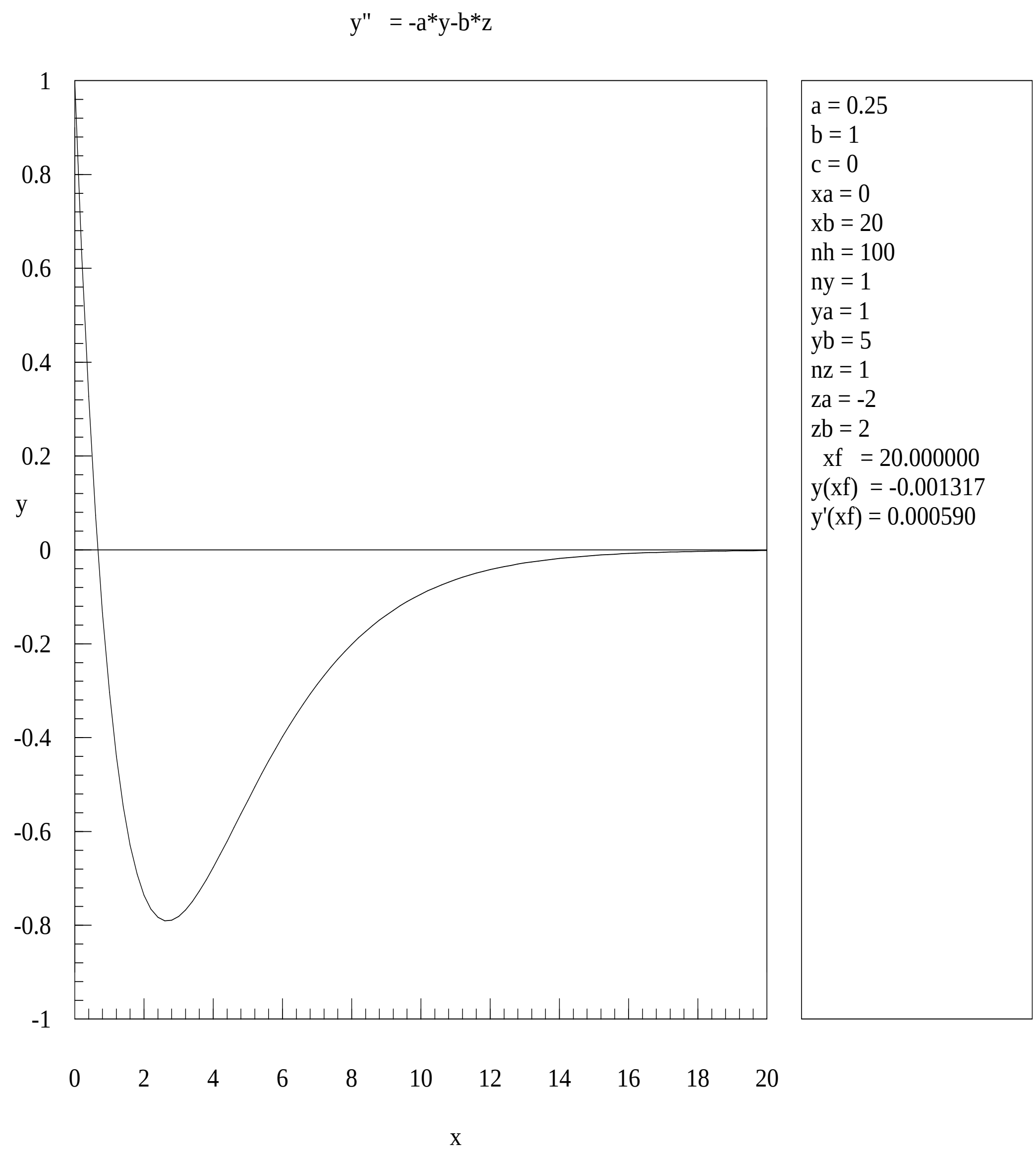

Figure 1. The particular solution of the de $4y''(t) + 4y'(t) + y(t) = 0$ satisfying the IC $y(0) = 1,\ y'(0) = -2$. [This graph was produced by the program **de**.]