## 3.1 Solutions

**1**. $(a)$ and $(b)$ are straightforward applications of the chain rule.

$(c)$ Since $v(x,t) = f(x,t)u(x/t, -1/t)$, where $f(x,t) = \frac{1}{\sqrt{t}}\exp(-x^2/4kt)$ and $f_t = kf_{xx}$,

$$v_t = f_t u + f \cdot [u_x \left(\frac{x}{t}\right)_t + u_t \left(-\frac{1}{t}\right)_t] = f_t u + f \cdot \left(-u_x \frac{x}{t^2} + u_t \frac{1}{t^2}\right).$$

$$\text{Moreover,} \quad v_x = f_x u + f u_x \frac{1}{t} \quad \text{and} \quad v_{xx} = f_{xx} u + 2 f_x u_x \frac{1}{t} + f u_{xx} \frac{1}{t^2}.$$

$$\begin{aligned} \text{Thus,} \quad v_t - k v_{xx} &= f_t u + f \cdot \left(-u_x \frac{x}{t^2} + u_t \frac{1}{t^2}\right) - k \left(f_{xx} u + 2 f_x u_x \frac{1}{t} + f u_{xx} \frac{1}{t^2}\right) \\ &= -f u_x \frac{x}{t^2} - 2k f_x u_x \frac{1}{t} = -u_x \frac{1}{t^2}(xf + 2kt f_x) = 0, \text{ since } \quad f_x = -\frac{x}{2kt} f. \end{aligned}$$

**2**. Follow the procedure of the derivation leading up to Proposition 1.

**3**. $(a)$ $u(x,t) = 4e^{-(2\pi/3)^2 2t} \sin(\frac{2\pi x}{3}) - e^{-(5\pi/3)^2 2t} \sin(\frac{5\pi x}{3})$
$(b)$ $u(x,t) = 5e^{-32\pi^2 t} \sin(4\pi x) + 2e^{-200\pi^2 t} \sin(10\pi x)$
$(c)$ $u(x,t) = \frac{3}{4} e^{-2(\pi/3)^2 t} \sin(\frac{\pi x}{3}) - \frac{1}{4} e^{-2\pi^2 t} \sin(\pi x)$
$(d)$ $u(x,t) = 9e^{-2(\pi/3)^2 t} \sin(\frac{\pi x}{3})$
$(e)$ $u(x,t) = 6e^{-(128\pi^2 t/9)} \sin(\frac{8\pi x}{3}) + e^{-50\pi^2 t} \sin(5\pi x)$.

**6**. $(a)$ $u(x,t) = 5e^{-t} \cos(x) + 3e^{-64t} \sin(8x)$
$(b)$ $u(x,t) = \frac{1}{2} + e^{-4t} (\cos(2x) - 6\sin(2x))$
$(c)$ $u(x,t) = \frac{1}{2} \left(9 + e^{-36t} \cos(6x)\right)$
$(d)$ $u(x,t) = 6e^{-t} \sin(x) - 7e^{-9t} (\cos(3x) + \sin(3x))$
$(e)$ $u(x,t) = \frac{1}{2} \left(5 + 3e^{-4t} \cos(2x) + 4e^{-4t} \sin(2x)\right)$.

**12**. (*a*) We have

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t}\left(u_1 + (u_0 - u_1)\frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{2}x/\sqrt{kt}} e^{-y^2}\,dy\right) = \frac{2}{\sqrt{\pi}}\exp\left(-\frac{x^2}{4kt}\right)\frac{\partial}{\partial t}\left(\tfrac{1}{2}x(kt)^{-\frac{1}{2}}\right) \\ &= \frac{2}{\sqrt{\pi}}\exp(-\frac{x^2}{4kt})\left(-\tfrac{1}{4}x(kt)^{-\frac{3}{2}}k\right) = -\frac{k}{2\sqrt{\pi}}\exp(-\frac{x^2}{4kt})x(kt)^{-\frac{3}{2}}. \end{aligned}$$

$$\begin{aligned} \text{Also,}\quad u_x(x,t) &= \frac{\partial}{\partial x}\left(u_1 + (u_0 - u_1)\frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{2}x/\sqrt{kt}} e^{-y^2}\,dy\right) \\ &= \frac{2}{\sqrt{\pi}}\exp\left(-\frac{x^2}{4kt}\right)\frac{\partial}{\partial x}\left(\tfrac{1}{2}x(kt)^{-\frac{1}{2}}\right) \\ &= \frac{1}{\sqrt{\pi}}\exp\left(-\frac{x^2}{4kt}\right)(kt)^{-\frac{1}{2}}. \end{aligned}$$

$$\text{Thus,}\quad u_{xx}(x,t) = \frac{1}{\sqrt{\pi}}\exp(-\frac{x^2}{4kt})(-\frac{2x}{4kt})(kt)^{-\frac{1}{2}} = -\frac{1}{2\sqrt{\pi}}\exp(-\frac{x^2}{4kt})x(kt)^{-\frac{3}{2}}.$$

Hence, $u_t = ku_{xx}$.

(*b*) Note that for $x > 0$, $\lim_{t\to 0}\frac{x}{\sqrt{kt}} = \infty$. Thus,

$$\begin{aligned} &\lim_{t\to 0}\left(u_1 + (u_0 - u_1)\frac{2}{\sqrt{\pi}}\int_0^{\frac{1}{2}x/\sqrt{kt}} e^{-y^2}dy\right) = u_1 + (u_0 - u_1)\frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-y^2}dy \\ = \;& u_1 + (u_0 - u_1)\frac{2}{\sqrt{\pi}}\frac{1}{2}\sqrt{\pi} = u_0. \end{aligned}$$