Math 473 PDE Solutions for A4

1. By Theorem 1, $\cos^3(x) = \frac{3}{4}\cos(x) + \frac{1}{4}\cos(3x)$ implies that $a_1 = \frac{3}{4}$ and $a_3 = \frac{1}{4}$. The integrals are πa_1 and πa_3, respectively.

2. (*a*) FS $f(x) = \frac{1}{8} - \frac{1}{8}\cos(4\pi x)$.
(*b*) FS $f(x) = \frac{1}{2}\cos(x) + \sin(x) - \frac{1}{2}\cos(3x)$.

7. (*a*) FS $f(x) = \frac{1}{\pi}(e^{\pi} - e^{-\pi})\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2}[\cos(nx) - n\sin(nx)]\right]$.
(*b*) FS $f(x) = \frac{1}{\pi}(e^{2\pi} - 1)\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{1+n^2}[\cos(nx) - n\sin(nx)]\right]$.

8. We have $a_0 = \frac{1}{\pi}\int_0^{\pi} \sin(x)\,dx = \frac{2}{\pi}$. For $n > 0$, we use Green's formula on the interval $[0,\pi]$:

$$\begin{aligned}
\pi a_n &= \int_0^{\pi} \sin(x)\cos(nx)\,dx = \langle c_n, s\rangle = -\frac{1}{n^2}\langle c_n'', s\rangle \\
&= -\frac{1}{n^2}\left(c_n'(x)s(x) - c_n(x)s'(x)|_0^{\pi} + \langle c_n, s''\rangle\right) = -\frac{1}{n^2}\left((-1)^n + 1 - \langle c_n, s\rangle\right) \\
&= -\frac{1}{n^2}\left((-1)^n + 1 - \pi a_n\right).
\end{aligned}$$

Thus, $\pi a_n\left(1 - \frac{1}{n^2}\right) = -\frac{1}{n^2}\left((-1)^n + 1\right)$ or $a_n = \frac{-1}{\pi}\frac{(-1)^n+1}{n^2-1}$. A similar calculation yields
$\pi b_n\left(1 - \frac{1}{n^2}\right) = 0$ or $b_n = 0$ for $n > 1$. For $n = 1$, we get $b_1 = \frac{1}{\pi}\int_0^{\pi} \sin^2(x)\,dx = \frac{1}{2}$. Thus,

$$\begin{aligned}
\text{FS } f(x) &= \tfrac{1}{\pi} + \tfrac{1}{2}\sin(x) - \tfrac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n + 1}{n^2 - 1}\cos(nx) \\
&= \tfrac{1}{\pi} + \tfrac{1}{2}\sin(x) - \tfrac{2}{\pi}\left(\tfrac{1}{1\cdot 3}\cos(2x) + \tfrac{1}{3\cdot 5}\cos(4x) + \ldots\right).
\end{aligned}$$

9. (*a*) $b_n = 0$, $a_0 = 2\int_0^1 \left(x^4 - 2x^2 + 1\right)\ dx = 2\left(\frac{1}{5}x^5 - \frac{2}{3}x^2 + x\right)\Big|_0^1 = 2\cdot\frac{3-10+15}{15} = \frac{16}{15}$.

$a_n = \int_{-1}^{1}(x^2-1)^2\cos(n\pi x)dx = \langle c_n, (x^2-1)^2\rangle = -\left(\frac{1}{n\pi}\right)^2\langle c_n'', (x^2-1)^2\rangle$

$= -\left(\frac{1}{n\pi}\right)^2\left(\left[c_n'(x)(x^2-1)^2 - c_n(x)(x^2-1)2x\right]\Big|_{-1}^{1} + \langle c_n, 12x^2 - 4\rangle\right)$

$= -\left(\frac{1}{n\pi}\right)^2\left(0 + \langle c_n, 12x^2 - 4\rangle\right) = \left(\frac{1}{n\pi}\right)^4\langle c_n'', 12x^2 - 4\rangle$

$= \left(\frac{1}{n\pi}\right)^4\left(\left[c_n'(x)(12x^2-4) - c_n(x)24x\right]\Big|_{-1}^{1} + \langle c_n, 24\rangle\right)$

$= \left(\frac{1}{n\pi}\right)^4\left(-c_n(x)24x\Big|_{-1}^{1} + 0\right) = -48\left(\frac{1}{n\pi}\right)^4(-1)^n$.

Thus, FS $f(x) = \frac{8}{15} - \frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}(-1)^n\cos(n\pi x)$.

(*b*) $M = \max_{-1\le x\le 1}|f''(x)| = \max_{-1\le x\le 1}|12x^2 - 4| = 8$. Using Theorem 2,

$|f(x) - S_N(x)| \le \frac{4L^2M}{\pi^2 N}$. Thus, we set $\frac{4L^2M}{\pi^2N} \le .001 \Rightarrow N \ge 4\cdot 8\cdot 1000\pi^{-2} \approx 3242.28$.

(*c*) $|\text{FS } f(x) - S_N(x)| = \left|\frac{48}{\pi^4}\sum_{n=N+1}^{\infty}\frac{(-1)^n}{n^4}\cos(n\pi x)\right| \le \frac{48}{\pi^4}\int_N^{\infty} x^{-4}\,dx = \frac{16}{\pi^4}N^{-3} < .001$

$\Rightarrow N \ge \left(\frac{16000}{\pi^4}\right)^{\frac{1}{3}} \approx 5.48$. Thus, $N = 6$ suffices.

(*d*) We put $x = 1$ in $(x^2-1)^2 =$FS $f(x) = \frac{8}{15} - \frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\cos(n\pi x)$, obtaining

$\frac{48}{\pi^4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\cos(n\pi) = \frac{8}{15}$, or $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$.

Section 4.2

3. Use Theorem β and recall that FS $f(x)$ is periodic of period 2 and converges to the average at jumps. One need not compute FS $f(x)$ to draw its graph.

9. (a) Using the distributive law, $(\mathbf{a}+r\mathbf{b})\cdot(\mathbf{a}+r\mathbf{b}) = (\mathbf{a}+r\mathbf{b})\cdot\mathbf{a}+(\mathbf{a}+r\mathbf{b})\cdot(r\mathbf{b})$, etc.. Since squares are nonnegative, $h(r) = \|\mathbf{a}+r\mathbf{b}\|^2 \geq 0$ for all r.

(b) For $\mathbf{b} \neq \mathbf{0}$, $h(r)$ is a quadratic function of r, and hence the graph of $h(r)$ is a parabola. Since $h(r) \geq 0$, the parabola will contact the r-axis in at most one point.

(c) Upon solving $h(r) = 0$ using the quadratic formula, the quantity $[2\mathbf{a}\cdot\mathbf{b}]^2 - 4(\mathbf{b}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{a})$ appearing under the square root would have to be positive or else there would be two real roots.

10. Since the properties of dot products which were used in Problem 9, also hold for inner products $(\langle , \rangle)$ of functions, the same proof works.