Math 473 PDE Solutions for A5

Section 4.3

3. If $f(-x) = f(x)$, then using the chain rule $f'(-x)(-1) = f'(x)$ (i.e., $f'(x)$ is odd), etc..

9. (a) Since $f(x)$ is even about $x = \frac{1}{2}$, we have $b_n = 0$ for n even, and

$$\begin{aligned} b_{2m+1} &= 4\int_0^{\frac{1}{2}} x \sin((2m+1)\pi x)\, dx \\ &= 4\left(-x \frac{\cos((2m+1)\pi x)}{(2m+1)\pi}\Big|_0^{\frac{1}{2}} + \int_0^{\frac{1}{2}} \frac{\cos((2m+1)\pi x)}{(2m+1)\pi}\, dx\right) \\ &= 4\, \frac{\sin((2m+1)\pi x)}{((2m+1)\pi)^2}\Big|_0^{\frac{1}{2}} = \frac{4(-1)^m}{[(2m+1)\pi]^2}. \end{aligned}$$

Thus, $\quad u(x,t)=\dfrac{4}{\pi^2}\displaystyle\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}e^{-(2m+1)^2\pi^2 t}\sin\left((2m+1)\pi x\right).$

(b) The initial heat energy outflux through the endpoints of $[\frac{1}{2}-\epsilon,\frac{1}{2}+\epsilon]$ is proportional to $u_x(\frac{1}{2}-\epsilon,0)-u_x(\frac{1}{2}+\epsilon,0)=1-(-1)=2=-\left(\frac{d}{dt}\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon}u(x,t)\,dx\right)_{t=0^+}\approx -\left(\frac{d}{dt}2\epsilon u(\frac{1}{2},t)\right)_{t=0^+}=-2\epsilon u_t(\frac{1}{2},0^+)$ for small ϵ, if we were to assume that $u_t(\frac{1}{2},0^+)$ exits. Thus, $-\epsilon u_t(\frac{1}{2},0^+)\approx 1$ regardless of the size of ϵ, which is impossible. Thus, $u_t(\frac{1}{2},0^+)$ cannot exist.

10. (a) The B.C. dictate that $f(x)$ needs to be represented by a series of the form
$\sum_{n=0}^{\infty}c_n\sin[(n+\frac{1}{2})x]$. Thus, by Theorem 3,

$$\begin{aligned} c_n &= \frac{2}{\pi}\int_0^{\pi}f(x)\sin\left((n+\tfrac{1}{2})x\right)\,dx \\ &= \frac{2}{\pi}\int_{\pi/2}^{\pi}\sin\left((n+\tfrac{1}{2})x\right)\,dx=-\frac{2}{\pi}\left.\frac{\cos\left((n+\frac{1}{2})x\right)}{n+\frac{1}{2}}\right|_{\pi/2}^{\pi} \\ &= \frac{2}{\pi}\frac{\cos\left((n+\frac{1}{2})\frac{\pi}{2}\right)}{n+\frac{1}{2}}. \end{aligned}$$

Thus, the formal solution is

$$u(x,t)=\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos\left((n+\frac{1}{2})\frac{\pi}{2}\right)}{n+\frac{1}{2}}e^{-(n+\frac{1}{2})^2kt}\sin\left((n+\frac{1}{2})x\right).$$

(b) Note that $u(\frac{\pi}{2},0)=\frac{1}{2}[f(\frac{\pi}{2}^+)+f(\frac{\pi}{2}^-)]=\frac{1}{2}\neq 1=f(\frac{\pi}{2})$. Moreover, for any N, we can find $\epsilon>0$ such that $u_n(\frac{\pi}{2}+\epsilon,0)<\frac{2}{3}$ while $f(\frac{\pi}{2}+\epsilon)=1$. Thus, $|u_n(\frac{\pi}{2}+\epsilon,0)-f(\epsilon)|>\frac{1}{3}$.

Section 4.4

7. (a) $\exp\left(\int\frac{x-2x}{x^2}\,dx\right)=\exp\left(\int-\frac{1}{x}\,dx\right)=\frac{1}{x}$, $(xy')'+\frac{x^2-m^2}{x}y=0$.

(b) $\exp\left(\int\frac{-2x+2x}{1-x^2}\,dx\right)=1$, $\left((1-x^2)y'\right)'+m(m+1)y=0$.

(c) $\exp\left(\int\frac{-x+2x}{1-x^2}\,dx\right)=\exp\left(-\frac{1}{2}\int\frac{-2x}{1-x^2}\,dx\right)=\dfrac{1}{\sqrt{1-x^2}}$, $\left(\sqrt{1-x^2}y'\right)'+\dfrac{m^2}{\sqrt{1-x^2}}y=0$.

(d) $\exp\left(\int-2x\,dx\right)=e^{-x^2}$, $\left(e^{-x^2}y'\right)'+2me^{-x^2}y=0$.

(e) $\exp\left(\int\frac{(1-x)-1}{x}\,dx\right)=\exp\left(\int-1\,dx\right)=e^{-x}$, $(xe^{-x}y')'+my=0$.

10. This is a direct consequence of Theorem 5. Alternatively,

$$(m(m+1)-n(n+1))\int_{-1}^{1} y_n(x)y_m(x)\,dx$$
$$=\int_{-1}^{1} y_n(x)m(m+1)y_m(x)\,dx-\int_{-1}^{1} n(n+1)y_n(x)y_m(x)\,dx$$

$$=-\int_{-1}^{1} y_n(x)\left((1-x^2)y_m'(x)\right)'\,dx+\int_{-1}^{1}\left((1-x^2)y_n'(x)\right)'y_m(x)\,dx$$
$$=\int_{-1}^{1} y_n'(x)(1-x^2)y_m'(x)\,dx-\int_{-1}^{1}(1-x^2)y_n'(x)y_m'(x)\,dx=0.$$

13. Let $y(x)$ be an eigenfunction. Then

$$-\lambda\int_0^1 y(x)^2\,dx=\int_0^1 y''(x)y(x)\,dx=y'(x)y(x)|_0^1-\int_0^1 y'(x)^2\,dx=-y(1)^2-\int_0^1[y'(x)]^2dx<0,$$

since $y(x)$ is not constant (otherwise $y(x)\equiv 0$ by the first B.C.). Thus, $\lambda\int_0^1 y(x)^2dx>0$, and so $\lambda>0$.