Math 473 Assignment 6

Professor:	*Richard Hall*
Instructions:	*Please make use of a computer-algebra program such as Mathematica, and explain your solutions carefully.*
Due Date:	*9th March 2017.*

Consider the Rayleigh-Ritz theorem and Sturm-Liouville problem $Hy = -y'' = \lambda y$, with BC $y(0) = y(a) = 0$, and inner product $(f, g) = \int_0^a f(x)g(x)dx$. The exact eigenvalues of this problem we have found to be $\lambda_n = (n\pi/a)^2$, $n = 1, 2, 3, \ldots$ If we look at H in an N-dimensional linear space spanned by the o.n. functions $\{\phi_n\}_{n=1}^N$ satisfying the BC, then the first N eigenvalues of the SL problem are bounded above, one by one, by the eigenvalues of the $N \times N$ matrix H with elements $H_{ij} = (\phi_i, H\phi_j)$.

6.1 Estimate λ_1 by using the trial function $\phi_1(x) = c_1p_1(x)$, where $p_1(x) = x(a - x)$, and the normalization constant c_1 is chosen so that $\|\phi_1\| = 1$. Thus we look at H in a one-dimensional space.

6.2 Note that $p_1(x)$ is even about $x = a/2$. Now consider the polynomial function $p_2(x) = p_1(x)(a - 2x)$ and its normalized partner $\phi_2(x) = c_2p_2(x)$. These functions are odd about $x = a/2$. Verify that $(p1, p2) = 0$, and explain why this is so by a symmetry argument.

6.3 Define the matrix H by its matrix elements $H_{ij} = (\phi_i, H\phi_j)$. Show that $H_{ij} = (\phi_i', \phi_j')$ for the present problem. Show also that $H_{12} = H_{21} = 0$. Thus the 2×2 matrix H is already diagonal and the diagonal elements immediately provide upper eigenvalue bounds given by $\lambda_1 \le H_{11}$, (found above in (6.1)) and $\lambda_2 \le H_{22}$. Now find the bound H_{22}, and compare the result with the known exact eigenvalues for the case $a = 2$.

6.4 Devise a third polynomial $p_3(x)$ that satisfies the BC and is orthogonal to p_1 and p_2. HINT: construct an even polynomial (about $x = a/2$) of degree 4. This is orthogonal to p_2, by symmetry. Now subtract $bp_1(x)$ and adjust b so that p_3 is orthogonal to p_1. Now normalize with a constant factor c_3 to yield $\phi_3 = c_3p_3$, and $\|\phi_3\| = 1$.

6.5 Consider the 3-dimensional space $\{\phi_i\}_{i=1}^3$ and construct the 3×3 symmetric matrix with elements H_{ij}. For the case $a = 2$ find the eigenvalues of H and compare these with the results of (6.3) and the known exact values.