Math 473 Assignment-7 Solutions

5.1 Solutions

1. (a) $u(x,t) = 3\cos(\frac{\pi a t}{L})\sin(\frac{\pi x}{L}) + \frac{L}{4\pi a}\sin(\frac{2\pi a t}{L})\sin(\frac{2\pi x}{L}) - \cos(\frac{4\pi a t}{L})\sin(\frac{4\pi x}{L})$
(b) $u(x,t) = \frac{3}{4}\cos(\frac{\pi a t}{L})\sin(\frac{\pi x}{L}) - \frac{1}{4}\cos(\frac{3\pi a t}{L})\sin(\frac{3\pi x}{L})$
(c) $u(x,t) = \frac{L}{4\pi a}\sin(\frac{\pi a t}{L})\sin(\frac{\pi x}{L}) + \frac{L}{12\pi a}\sin(\frac{3\pi a t}{L})\sin(\frac{3\pi x}{L})$
(d) $u(x,t)$ is the sum of the solutions for parts (b) and (c).

2. (a) and (b) are straightforward.
(c) $u_t = \bar{u}_{\bar{t}}\bar{t}_t + \bar{u}_{\bar{x}}\bar{x}_t$, and

$$\begin{aligned} u_{tt} &= [\bar{u}_{\bar{t}\bar{t}}\bar{t}_t + \bar{u}_{\bar{t}\bar{x}}\bar{x}_t]\bar{t}_t + \bar{u}_{\bar{t}}\bar{t}_{tt} + [\bar{u}_{\bar{x}\bar{x}}\bar{x}_t + \bar{u}_{\bar{x}\bar{t}}\bar{t}_t]\bar{x}_t + \bar{u}_{\bar{x}}\bar{x}_{tt} \\ &= \bar{u}_{\bar{t}\bar{t}}\bar{t}_t\bar{t}_t + 2\bar{u}_{\bar{t}\bar{x}}\bar{x}_t\bar{t}_t + \bar{u}_{\bar{x}\bar{x}}\bar{x}_t\bar{x}_t + \bar{u}_{\bar{t}}\bar{t}_{tt} + \bar{u}_{\bar{x}}\bar{x}_{tt} \\ &= \bar{u}_{\bar{t}\bar{t}}\cosh^2(\omega) + 2\bar{u}_{\bar{t}\bar{x}}a\sinh(\omega)\cosh(\omega) + \bar{u}_{\bar{x}\bar{x}}a^2\sinh^2(\omega). \end{aligned}$$

Similarly,

$$\begin{aligned} u_{xx} &= \bar{u}_{\bar{t}\bar{t}}\bar{t}_x\bar{t}_x + 2\bar{u}_{\bar{t}\bar{x}}\bar{x}_x\bar{t}_x + \bar{u}_{\bar{x}\bar{x}}\bar{x}_x\bar{x}_x + \bar{u}_{\bar{t}}\bar{t}_{xx} + \bar{u}_{\bar{x}}\bar{x}_{xx} \\ &= \bar{u}_{\bar{t}\bar{t}}a^{-2}\sinh^2(\omega) + 2\bar{u}_{\bar{t}\bar{x}}a^{-1}\sinh(\omega)\cosh(\omega) + \bar{u}_{\bar{x}\bar{x}}\cosh^2(\omega) \end{aligned}$$

Hence,

$$\begin{aligned} u_{tt} - a^2u_{xx} &= \bar{u}_{\bar{t}\bar{t}}\cosh^2(\omega) + 2\bar{u}_{\bar{t}\bar{x}}a\sinh(\omega)\cosh(\omega) + \bar{u}_{\bar{x}\bar{x}}a^2\sinh^2(\omega) \\ &\quad -a^2[\bar{u}_{\bar{t}\bar{t}}a^{-2}\sinh^2(\omega) + 2\bar{u}_{\bar{t}\bar{x}}a^{-1}\sinh(\omega)\cosh(\omega) + \bar{u}_{\bar{x}\bar{x}}\cosh^2(\omega)] \\ &= \bar{u}_{\bar{t}\bar{t}}[\cosh^2(\omega) - \sinh^2(\omega)] - a^2\bar{u}_{\bar{x}\bar{x}}[\cosh^2(\omega) - \sinh^2(\omega)] = \bar{u}_{\bar{t}\bar{t}} - a^2\bar{u}_{\bar{x}\bar{x}}. \end{aligned}$$

5.2 Solutions

1. (a) $\frac{1}{2}(1+\frac{1}{2a})(x+at)^2 + \frac{1}{2}(1-\frac{1}{2a})(x-at)^2$ (b) $\exp\left(-(x-at)^2\right)$
(c) t (d) 1
(e) $\sin(x+at)$ (f) $\frac{1}{2}t - \frac{1}{4a}\cos(2x)\sin(2at)$.

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3. (*a*) $u(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(r)\,dr$. Thus,

$$
\begin{aligned}
u(-x,t) &= \tfrac{1}{2}[f(-x+at) + f(-x-at)] + \frac{1}{2a}\int_{-x-at}^{-x+at} g(r)\,dr \\
&= \tfrac{1}{2}[f(-(-x+at)) + f(-(x+at))] - \frac{1}{2a}\int_{x+at}^{x-at} g(-s)\,ds \\
&= \tfrac{1}{2}[f(x-at) + f(x+at)] - \frac{1}{2a}\int_{x+at}^{x-at} g(s)\,ds \\
&= \tfrac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(s)\,ds = u(x,t).
\end{aligned}
$$

(*b*) $v_{tt}(x,t) = u_{tt}(-x,t) = a^2 u_{xx}(-x,t) = a^2(-1)^2 u_{xx}(-x,t) = a^2 v_{xx}(x,t)$, and $v(x,0) = u(-x,0) = f(-x) = f(x)$ and $v_t(x,0) = u_t(-x,0) = g(-x) = g(x)$.

(*c*) Since $v(x,t)$ satisfies the wave equation and the same I.C. as $u(x,t)$, we must have $v(x,t) = u(x,t)$ by uniqueness (cf. Theorem 1).