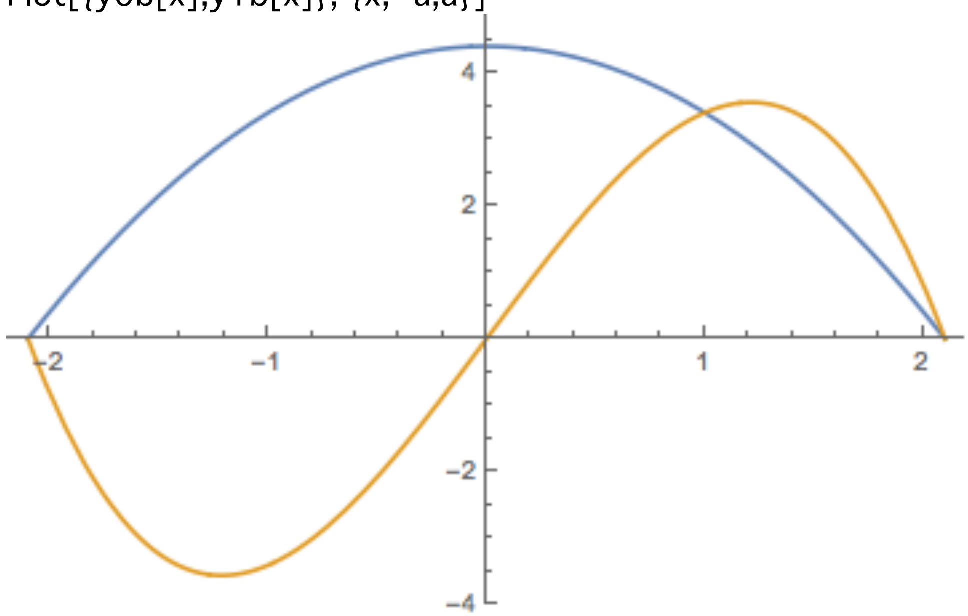
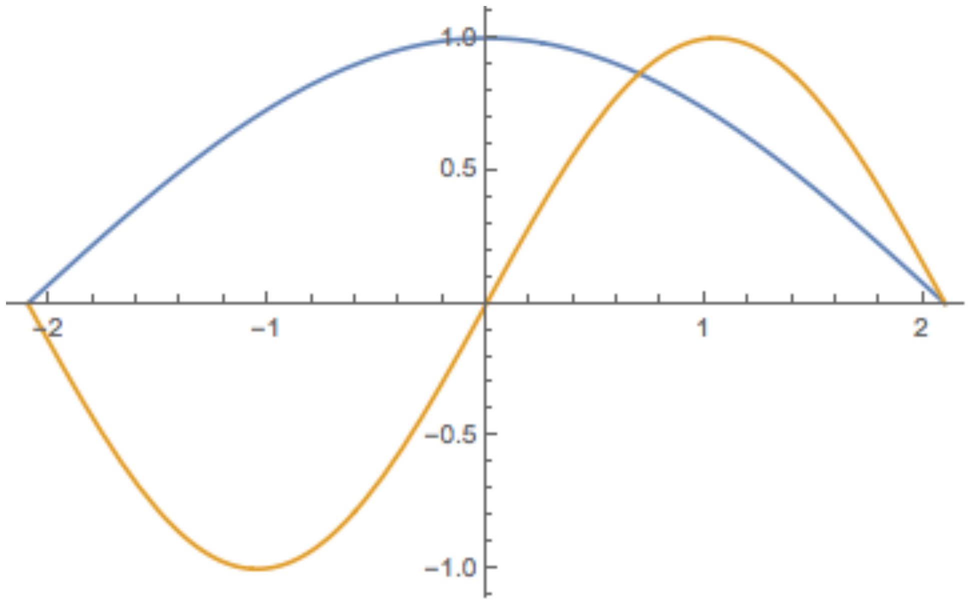
```
(* ------------------------------------------- *)
(*   a473-8s.nb   Richard Hall                *)
(* ------------------------------------------- *)
 Clear["Global`*"]
 (* some trial wave functions  *)
 (*
 global box size  a
This use of a global parameter is not the most elegant
code style but it allows the partial wrt  x  to be effected
simply by the use of the prime.
 *)
 y0a[x_] := Cos[Pi x/(2 a)]
 y0b[x_]  :=a^2-x^2
 y1a[x_ ]:= Sin[Pi x/ a]
 y1b[x_] := x y0b[x]
 (* give a value to global  a  for immediate use  *)
 a = 2.1
 2.1
 (* plot the unnormalized trial functions *)
 Plot[{y0b[x],y1b[x]}, {x, -a,a}]
```



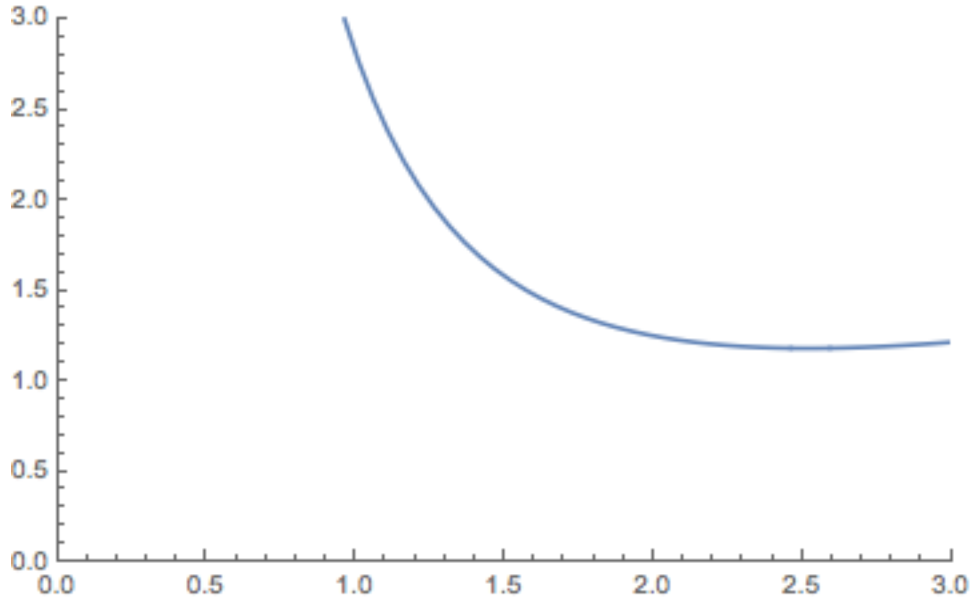

```
 Plot[{y0a[x],y1a[x]},{x,-a,a}]
```



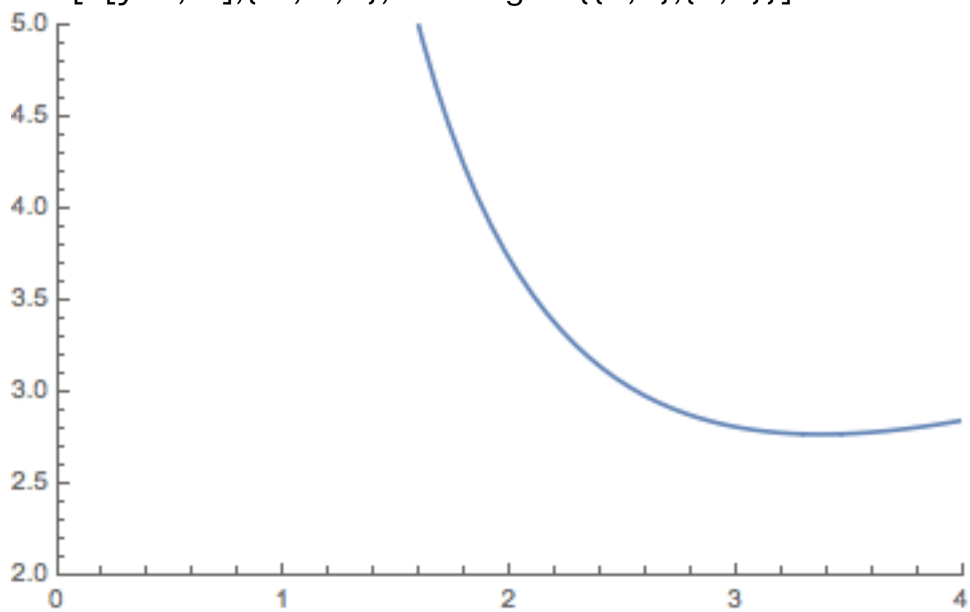

```
 (* example of the linear potential v(x) = |x|  *)
 v[x_] := Abs[x]
 (* inner product *)
 ip[f_,g_] := Integrate[f[x] g[x],{x,-a,a}]
 normc[f_] := ip[f,f]^(-1/2)
 (* Energy as a function of a (divide by ||y||^2)   *)
 e[y_,a1_]:= Block[{},a := a1; (*  a is a global parameter
*)N[Integrate[y'[x]^2 + v[x] y[x]^2,{x,-a,a}]/Integrate[y[x]^2,{x,-a,a}]]]
 (*
  Minimize the upper estimates for e0 and e1  wrt a.
Note that we may look at the even and odd subspaces separately:
 by symmetry, the 2x2 matrix  (i|H|j) is diagonal for i,j in {0,1}.
First we try the polynomials y0b and y1b.

NB  There are two distinct problems here regarding h = -D^2 +|x|:
    (1) The eigenvalues of h in L^2[-a1,a1]ie h is confined in [-a1,a1].
     For this problem we could use say {y0b, y1b} with a ≤ a1.
(2) The eigenvalues of h in L^2[R], ie a = infinity.
       For this we use any a and optimize the aprrox energies wrt a.
      In these notes we explore (2) in the odd and even states separately,
getting upper bounds for the bottom in each subspace.
     First we see if there indeed are minima to search for.
*)


 Plot[e[y0b,as],{as, 0.5,3},PlotRange->{{0,3},{0,3}}]
```



FindMinimum[e[y0b,as],{as,2}] (* 2 is just a starting value *)
{1.18118,{as->2.51984}}
Plot[e[y1b,as],{as, 1,4},PlotRange->{{0,4},{2,4}}]



FindMinimum[e[y1b,as],{as,2}]
{2.76751,{as->3.37373}}
(*
Now we look at the results for the trig trial functions
which turn out to do better than the polnomials.
Accurate numerical values are e0 = 1.01879 and e1 = 2.33811.
The analytical solution is in terms of Airy functions.
In QM books the problem is tagged by 'linear potential'.

```
*)
 FindRoot[AiryAiPrime[x] ,{x, -1}]
{x->-1.01879}
FindRoot[AiryAi[x],{x,-3}]
{x->-2.33811}
FindMinimum[e[y0a,as],{as,2}]
{1.13772,{as->2.55072}}
FindMinimum[e[y1a,as],{as,2}]
{2.55377,{as->3.40502}}
(* The trig results are better  *)
```

Math 473 Assignment-8 Solutions for 9.1 {13, 14}

13.

$$u(x,y,t) = \sum_{n,m=1}^{\infty} c_n c_m \frac{1}{\pi a\sqrt{n^2+m^2}} \sin\left(\pi a\sqrt{n^2+m^2}\, t\right) \sin(n\pi x)\sin(m\pi y),$$

$$\begin{aligned}
\text{where} \quad c_k &= 2\int_0^1 z(z-1)\sin(k\pi z)\,dz = 2\langle s_k, z(z-1)\rangle = \frac{-2}{(k\pi)^2}\langle s_k'', z(z-1)\rangle \\
&= \frac{-2}{(k\pi)^2}\left((s_k'(z)z(z-1) - s_k(z)(2z-1))\big|_0^1 + \langle s_k, 2\rangle \right) = \frac{-2}{(k\pi)^2}\langle s_k, 2\rangle \\
&= \frac{4}{(k\pi)^3}\cos(k\pi z)\big|_0^1 = \frac{4}{(k\pi)^3}\left(\cos(k\pi) - 1\right) = \frac{4}{(k\pi)^3}\left((-1)^k - 1\right) \\
&= \begin{cases} -\left(\frac{2}{k\pi}\right)^3 & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even.} \end{cases}
\end{aligned}$$

Thus,

$$u(x,y,t) = \sum_{n,m=1 \text{ (odd)}}^{\infty} \left(\frac{4}{nm\pi^2}\right)^3 \frac{1}{\pi a\sqrt{n^2+m^2}} \sin\left(\pi a\sqrt{n^2+m^2}\, t\right) \sin(n\pi x)\sin(m\pi y).$$

14. The frequencies of the drum are of the form $\frac{a}{2}\sqrt{\left(\frac{n}{L}\right)^2 + \left(\frac{m}{M}\right)^2}$, for $m, n = 1, 2, 3, \ldots$. The lowest frequency is $300\sqrt{\left(\frac{1}{L}\right)^2 + \left(\frac{1}{M}\right)^2} = 300$, and if $L \geq M$, the second lowest frequency is $300\sqrt{\left(\frac{2}{L}\right)^2 + \left(\frac{1}{M}\right)^2} = 400$. Thus, $\frac{3}{L^2} = \frac{16}{9} - 1 = \frac{7}{9}$, or $L = \sqrt{\frac{27}{7}}$. Moreover, $\frac{1}{M^2} = 1 - \frac{1}{L^2} = 1 - \frac{7}{27} = \frac{20}{27}$ or $M = \sqrt{\frac{27}{20}}$.