Math 473 A-9 Solutions

9.5 Solutions

1. Consider equations (1) and (2) with $c = 1$ and $f \equiv 0$. Then the idea in Problem 4(*b*) of Section 9.3 may be applied.

2. (*a*)

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{r_\circ} (g(r,\theta)\Delta f(r,\theta) - f(r,\theta)\Delta g(r,\theta))\, r\, dr\, d\theta \\
=& \int_0^{2\pi}\int_0^{r_\circ} \left(g\cdot\left(\frac{1}{r}(rf_r)_r + \frac{1}{r^2}f_{\theta\theta}\right) - f\cdot\left(\frac{1}{r}(rg_r)_r + \frac{1}{r^2}g_{\theta\theta}\right)\right) r\, dr\, d\theta \\
=& \int_0^{2\pi}\int_0^{r_\circ} \left(\left(g(rf_r)_r + \frac{1}{r}gf_{\theta\theta}\right) - \left(f(rg_r)_r + \frac{1}{r}fg_{\theta\theta}\right)\right) dr\, d\theta \\
=& \int_0^{2\pi}\int_0^{r_\circ} g(rf_r)_r - f(rg_r)_r\, dr\, d\theta + \int_0^{r_o}\int_0^{2\pi} \frac{1}{r}gf_{\theta\theta} - \frac{1}{r}fg_{\theta\theta}\, dr\, d\theta \\
=& \int_0^{2\pi}\int_0^{r_\circ} (grf_r)_r - g_r r f_r - (frg_r)_r + f_r r g_r\, dr\, d\theta - \int_0^{r_\circ}\int_0^{2\pi} \frac{1}{r}g_\theta f_\theta - \frac{1}{r}f_\theta g_\theta\, dr\, d\theta \\
=& \int_0^{2\pi}\int_0^{r_\circ} (grf_r)_r - (frg_r)_r\, dr = \int_0^{2\pi} grf_r - frg_r|_0^{r_\circ}\, d\theta \\
=& \int_0^{2\pi} (g(r_\circ,\theta)f_r(r_\circ,\theta) - f(r_\circ,\theta)g_r(r_\circ,\theta))\, r_\circ\, d\theta
\end{aligned}$$

3. (*a*) Set $f(r,\theta) = J_m(br)\sin(m\theta)$ and $g(r,\theta) = J_m(\beta r)\sin(m\theta)$. Note that $\Delta f = -b^2 f$ and $\Delta g = -\beta^2 g$, and apply 2(*a*), where, $\Delta f = f_{rr} + r^{-1}f_r + r^{-2}f_{\theta\theta}$. The integral with respect to θ introduces the same nonzero factor on each side.

(*b*) This is straightforword. Use the product rule on the left-hand side.

5. (*a*) Proceeding as in Example 1, we obtain

$$U(r,\theta,t) = \sum_{m=-\infty}^{\infty}\sum_{q=1}^{\infty} c_{m,q} \exp\left(-\left(\frac{j_{m,q}}{r_\circ}\right)^2 kt\right) J_m\left(\frac{j_{m,q}r}{r_\circ}\right) e^{im\theta},$$

where $$c_{m,q} = \frac{1}{\pi r_\circ^2 J_m'(j_{m,q})^2}\int_0^{2\pi}\int_0^{r_\circ} f(r,\theta) J_m\left(\frac{j_{m,q}r}{r_\circ}\right) e^{-im\theta} r\, dr\, d\theta$$

and $j_{m,q}$ is the q-th positive value of r such that $J_m(r) = 0$. We have used problem (3b).