Math 473 Solutions for Assignment 1

Professor: *Richard Hall*
Instructions: *Please explain your solutions carefully.*

1.1 Suppose $f(x)$ is (i) continuously differentiable for all $x \in \Re$, and (ii) the function f satisfies Cauchy's functional equation $f(x+y) = f(x) + f(y)$. Prove that $f(x) = f(1)x$.

Solution: $f(x+h) - f(x) = f(h)$. Thus $f'(x) = f'(0)$. Hence $f(x) = f'(0)x + c$. Since $f(0+0) = f(0) + f(0)$, therefore $f(0) = 0$, and $c = 0$. Meanwhile, $f(1) = f'(0)$. Thus $f(x) = f(1)x$.

1.1a Now solve Cauchy's equation under the weaker assumption that $f(x)$ is continuous. HINT: first solve for fractions, then recall that $x_n \to x \Rightarrow f(x_n) \to f(x)$.

Let m and n be positive integers. Since $f(n) = f(1 + n - 1) = f(1) + f(n-1)$, we see that $f(n) = nf(1)$. By similar reasoning we find $f(n/m) = (n/m)f(1)$. Since every real number can be expressed as the limit of a sequence $\{x_n\}$ of rational numbers, and $f(x)$ is continuous, therefore $f(x) = xf(1)$ for every real number x.

1.2 If $y = y(x)$, find the general solution to the ode $y' + 3y = \sin(x)$.

Solution: The integrating factor is $g(x) = e^{3x}$, and the general solution is
$y = Ce^{-3x} + \frac{1}{10}\left[3\sin(x) - \cos(x)\right]$.

1.3 If $u = u(x, y)$, find the general solution to the pde $u_x = 2x$.

Solution: By integrating over x, we have $u(x, y) = x^2 + C(y)$.

1.4 Find the general solution to the pde $u_x + 3u = \sin(x)$.

Solution: By adapting the solution to (1.2) we have
$u(x, y) = C(y)e^{-3x} + \frac{1}{10}\left[3\sin(x) - \cos(x)\right]$.

1.5 Suppose $u = u(x, y)$ satisfies the pde $3u_x + 4u_y + u = e^{x+y}$.

(a) Find the general solution.
(b) Find a particular solution satisfying $u(0, y) = e^y$.

Solution: The characteristic lines are determined by $x'(s) = 3$ and $y'(s) = 4$ to be $x(s) = 3s + x_o$ and $y(s) = 4s + y_o$. Thus $3y - 4x = 3y_o - 4x_o = \text{const}$. Along the characteristic lines we write $u(x, y) = v(s)$. Thus we have $v' + v = e^{7s + x_o + y_o}$. This is a first-order linear ode for $v(s)$ with general solution $v(s) = e^{-s}\left[\frac{1}{8}e^{8s + x_o + y_o} + C_1(3y - 4x)\right]$.
Many equivalent expressions are possible in terms of x and y, because $3y - 4x$ is constant. One possibility is:
$u(x, y) = e^{-x/3}C(3y - 4x) + \frac{1}{8}e^{x+y}$.
Thus $u(0, y) = C(3y) + \frac{1}{8}e^y$ which should equal e^y. Thus $C(X) = \frac{7}{8}e^{X/3}$, and we have
$u(x, y) = e^{-x/3}\frac{7}{8}e^{(3y-4x)/3} + \frac{1}{8}e^{x+y} = \frac{7}{8}e^{(3y-5x)/3} + \frac{1}{8}e^{x+y}$

1.6 Suppose $u = u(x, y)$ satisfies the pde $yu_x + u_y = 1$

(a) Find the general solution.
(b) Find a particular solution satisfying $u(0, y) = y + y^4$.

Solution: The characteristic curves are determined by $x'(s) = y$ and $y'(s) = 1$ to be $y(s) = s + y_o$ and $x(s) = s^2/2 + y_0 s + x_0$. Thus $x - \frac{1}{2}y^2 = x_0 - \frac{1}{2}y_o^2 = \text{const}$. On the characteristic curves we may write $u(x, y) = v(s)$, where $v'(s) = 1$. Thus $v(s) = s + C_1(x - \frac{1}{2}y^2)$. By substituting $s = y - y_0$, and absorbing y_0 in C_1 , we find $u(x, y) = y + C(x - \frac{1}{2}y^2)$. Thus $u(0, y) = y + C(-\frac{1}{2}y^2)$, which should equal $y + y^4$. This implies $C(X) = 4X^2$. Hence the particular solution sought is given by $u(x, y) = y + 4(x - \frac{1}{2}y^2)^2$.