Department of Mathematics & Statistics

Math 473 Mast 666 Mast 841 Sec A Final Exam March 2015

Partial Differential Equations

Professor: *Richard Hall*
Instructions: *Please answer all 5 questions which carry equal marks.*
Explain your work carefully.
Approved calculators are permitted.

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1. Consider the partial differential equation $yu_x(x, y) - 3u_y(x, y) + u(x, y) = y$.
   (a) Sketch the characteristic curves for this equation.
   (b) Find the general solution $u(x, y)$.
   (c) Find the particular solution satisfying the initial condition
   $u(x, 0) = \cosh(x)$

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2. Suppose that $u(x, t)$ represents the temperature in a side-insulated bar of length $L = 10$ which at time $t = 0$ has the temperature profile $u(x, 0) = f(x) = (70 - 5x)$. For times $t > 0$, the ends of the bar are insulated. Suppose that $u(x, t)$ satisfies the heat equation $u_t(x, t) = ku_{xx}(x, t) =$ where $k = \frac{1}{4}$.
   (a) Find the steady-state temperature profile $u(x, \infty)$.
   (b) Find the temperature profile $u(x, t)$ for $t > 0$.
   (c) If $Q(t) = \int_0^L u(x, t)dx$, show that $Q(t)$ is constant and explain what this means.

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3. Consider the Sturm-Liouville eigenproblem $-y''(x) + x^2y(x) = \lambda y(x)$ for $x \in [-1, 1]$ and with the boundary conditions BC: $y(-1) = y(1) = 0$.
   (a) What can you say about the set of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$?
   (b) Construct two suitable 'trial functions' $\phi_1(x), \phi_2(x)$, which are C^2 and satisfy the BC. Use these functions to estimate the first two eigenvalues λ_1, λ_2 .
   (c) Explain carefully what is the relationship between the estimates found in (b) and the unknown exact eigenvalues.

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4. Consider Laplace's equation $u_{xx} + u_{yy} = 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$ with the boundary conditions BC: $u(x,0) = -x + 4\sin(3\pi x)$, $u(0,y) = y + 7\sin(2\pi y)$, $u(x,1) = y + 2x$, and $u(1,y) = 4y - 1$.
   (a) Find a function of the form $w(x,y) = a + bx + cy + dxy$ such that $w(0,0) = 0$, $w(1,0) = -1$, $w(0,1) = 1$, $w(1,1) = 3$.
   (b) Write the solution $u(x,y)$ of (a) in the form $u(x,y) = v(x,y) + w(x,y)$, and solve for $u(x,y)$ by first finding $v(x,y)$.

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5. Consider the vibrations of a circular drum of radius r_0 whose displacement $u(r,\theta,t)$ satisfies the wave equation $c^2 \Delta u = u_{tt}$, where c is a constant. Suppose that the boundary of the drum is clamped so that $u(r_0,\theta,t) = 0$.
   (a) Find the differential equations satisfied by each of the factors in solutions of the form $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$.
   (b) Solve the equation for $\Theta(\theta)$.
   (c) Describe the normal modes of vibration of the drum.
