Math 473 Sec A Midterm Test 5^{th} March 2015

Professor: *Richard Hall*

Instructions: *Please answer all three questions.*
Explain your work clearly.
Duration: 1 hour.

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1. **[12]** Consider the partial differential equation given by

   $2u_x(x,y) + xu_y(x,y) + u(x,y) = e^x.$

   (a) What are the characteristic curves for this equation.

   (b) Find the general solution $u(x,y)$.

   (c) Find the particular solution satisfying the initial condition
   $u(0,y) = y^3.$

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2. **[12]** Suppose that $u(x,t)$ represents the temperature in a bar of length $L = 10$ which at time $t = 0$ has the temperature profile $u(x,0) = f(x) = 10.$ For times $t > 0,$ the ends of the bar are kept at constant temperatures given by $u(0,t) = T_1 = 10^o\,\mathrm{C}$ and $u(L,t) = T_2 = 40^o\,\mathrm{C}.$ Suppose that $u(x,t)$ satisfies the heat equation $u_{xx}(x,t) = k\,u_t(x,t),$ where $k = 2.$

   (a) Find the steady-state temperature profile $u(x,\infty)$.

   (b) Find an expression for the temperature profile $u(x,t)$ for $t > 0$.

   (c) Provide some qualitative sketches that show how the initial profile $u(x,0)$ evolves under the heat equation to the steady-state profile $u(x,\infty)$.

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3. **[6]** Consider the Sturm-Liouville eigenvalue problem on the interval $[a,b] \subset \Re,$ $a < b,$ given by $-(py')' + qy = \lambda g y,$ where the functions $p(x)$ and $g(x)$ are smooth and positive on (a,b), and $y(a) = y(b) = 0.$

   Please answer only either part (i) or part (ii) :

   (i) Find the eigenfunctions $\{y_n\}$ and eigenvalues $\{\lambda_n\}$ in the special case $q = 0$, $p = k^2$, and $g = 1$.

   (ii) In the general case suppose that $y_m(x)$ and $y_n(x)$ are eigenfunctions corresponding respectively to the eigenvalues λ_m and λ_n. Prove that if $\lambda_m \neq \lambda_n,$ then y_m is orthogonal to y_n with respect to the inner product defined by $(y_1, y_2) = \int_a^b y_1(x)\, y_2(x)\, g(x)\, dx.$