Math 473 Sec A Midterm Test Solution Notes March 2015

Professor: *Richard Hall*
Instructions: *Please answer all three questions.*
Explain your work clearly.
Duration: 1 hour.

1. **[12]** Consider the partial differential equation given by

   $2u_x(x,y) + xu_y(x,y) + u(x,y) = e^x$.

   (a) What are the characteristic curves for this equation.

   (b) Find the general solution $u(x,y)$.

   (c) Find the particular solution satisfying the initial condition

   $u(0,y) = y^3$.

Let $v(s) = u(x(s), y(s))$. The characteristic curves are obtained from $\{x'(s) = 2, y'(s) = x\}$ and then the pde on these curves becomes $v'(s) + v(s) = e^{x(s)}$. Solving for $x(s)$ and $y(s)$ we find $x = 2s + x_0$ and $y = s^2 + x_0 s + y_0$. Thus the curves are parabolas $y - x^2/4 = y_0 - x_0^2/4$. By solving the linear ode for $v(s)$ we then find $u = C(y - x^2/4)\exp(-x/2) + \exp(x)/3$. Fitting the IC (c) yields the function $C(X) = X^3 - 1/3$; thus the particular solution sought is $u = \left((y - x^2/4)^3 - 1/3\right)\exp(-x/2) + \exp(x)/3$.

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2. **[12]** Suppose that $u(x,t)$ represents the temperature in a bar of length $L = 10$ which at time $t = 0$ has the temperature profile $u(x,0) = f(x) = 10$. For times $t > 0$, the ends of the bar are kept at constant temperatures given b $u(0,t) = T_1 = 10\,^o\mathrm{C}$ and $u(L,t) = T_2 = 40\,^o\mathrm{C}$. Suppose that $u(x,t)$ satisfies the heat equation $u_{xx}(x,t) = k\,u_t(x,t)$, where $k = 2$.

   (a) Find the steady-state temperature profile $u(x,\infty)$.

   (b) Find an expression for the temperature profile $u(x,t)$ for $t > 0$.

   (c) Provide some qualitative sketches that show how the initial profile $u(x,0)$ evolves under the heat equation to the steady-state profile $u(x,\infty)$.

This is a very standard problem. $T_1 = 10,\ T_2 = 40,\ L = 10$,
$u(x,\infty) = T_1 + (T_2 - T_1)x/L = 10 + 3x$. $u(x,t) = v(x,t) + u(x,\infty)$.
Then $v(0,t) = v(L,t) = 0$, and $v(x,0) = f(x) = -3x$. We find the Fourier coefficients for a sine series for $f(x)$ are given by

$$b_n = \frac{2}{10}\int_0^{10} (-3x)\sin\left(\frac{n\pi x}{10}\right) dx = (-1)^n \frac{60}{n\pi}.$$

Thus

$$u(x,t) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right)\exp\left(-\frac{t}{2}\left(\frac{n\pi}{10}\right)^2\right) + 10 + 3x.$$

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3. **[6]** Consider the Sturm-Liouville eigenvalue problem on the interval $[a,b] \subset \Re$, $a < b$, given by $-(py')' + qy = \lambda g y$, where the functions $p(x)$ and $g(x)$ are smooth and positive on (a,b), and $y(a) = y(b) = 0$.

   Please answer only either part (i) or part (ii) :

   (i) Find the eigenfunctions $\{y_n\}$ and eigenvalues $\{\lambda_n\}$ in the special case $q = 0$, $p = k^2$, and $g = 1$.

   (ii) In the general case suppose that $y_m(x)$ and $y_n(x)$ are eigenfunctions corresponding respectively to the eigenvalues λ_m and λ_n. Prove that if $\lambda_m \neq \lambda_n$, then y_m is orthogonal to y_n with respect to the inner product defined by $(y_1, y_2) = \int_a^b y_1(x)\, y_2(x)\, g(x)\, dx$.

(i) By solving the ode $k^2 y'' = -\lambda y$ with the BC $y(a) = y(b) = 0$, one finds $\lambda_n = \left(\frac{n\pi k}{b-a}\right)^2$, and $y_n = \sin\left(\frac{n\pi(x-a)}{b-a}\right)$, where $n = 1, 2, 3, \ldots$. A very convenient approach is first to change variables to $L = (b-a)$ and $z = x - a$, which converts the given problem on $[a,b]$ to the more familiar one on $[0, L]$. It also comes out directly with a bit more algebraic effort with trigonometric identities.

(ii) One writes the two eigenequations, calling them (1) and (2). By integrating the difference $(1)y_1 - (2)y_2$ over the interval $[a,b]$ and using the BC, it follows $(y_1, y_2)(\lambda_1 - \lambda_2) = 0$. Green's SL formula (Text p265) can be used, or a direct approach using integration by parts: there is not much difference in the effort required.
