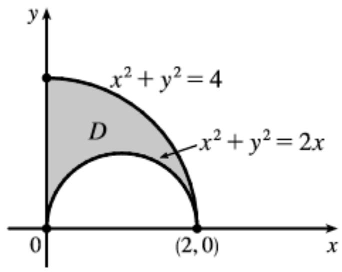
Mast 219 Midterm Test Solutions March 2016

(1)



$$\iint_D x\,dA = \iint_{\substack{x^2+y^2\le 4\\ x\ge 0,\, y\ge 0}} x\,dA \;-\; \iint_{\substack{(x-1)^2+y^2\le 1\\ y\ge 0}} x\,dA$$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2}\tfrac{1}{3}(8\cos\theta)\,d\theta - \int_0^{\pi/2}\tfrac{1}{3}(8\cos^4\theta)\,d\theta$$

$$= \tfrac{8}{3} - \tfrac{8}{12}\left[\cos^3\theta\sin\theta + \tfrac{3}{2}(\theta+\sin\theta\cos\theta)\right]_0^{\pi/2}$$

$$= \tfrac{8}{3} - \tfrac{2}{3}\left[0 + \tfrac{3}{2}\left(\tfrac{\pi}{2}\right)\right] = \frac{16-3\pi}{6}$$

(2) (a)

Given the sphere $x^2+y^2+z^2=4$, when $z=1$, we get $x^2+y^2=3$ so $D=\{(x,y)\mid x^2+y^2\le 3\}$ and $z=f(x,y)=\sqrt{4-x^2-y^2}$. Thus

$$A(S) = \iint_D \sqrt{[(-x)(4-x^2-y^2)^{-1/2}]^2+[(-y)(4-x^2-y^2)^{-1/2}]^2+1}\,dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}}\sqrt{\frac{r^2}{4-r^2}+1}\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}}\sqrt{\frac{r^2+4-r^2}{4-r^2}}\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}}\frac{2r}{\sqrt{4-r^2}}\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[-2(4-r^2)^{1/2}\right]_{r=0}^{r=\sqrt{3}}d\theta = \int_0^{2\pi}(-2+4)\,d\theta = 2\theta\Big]_0^{2\pi} = 4\pi$$

(2) (b)

The upper limit on r^2 is now 4 – h^2, and S(h) = 4 pi (2-h).
Thus, a slice of a sphere has the same surface area as an equally thick slice of a surrounding cylinder. If t is the thickness of a slice, the areas are both S = 4 pi t.
For a sphere of radius a, the surface areas are given by S = 2 pi a t.

(3)

Place the center of the base at $(0,0,0)$, then the density is $\rho(x,y,z)=Kz$, K a constant. Then

$m=\int_0^{2\pi}\int_0^{\pi/2}\int_0^a (K\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi K\int_0^{\pi/2}\cos\phi\sin\phi\cdot\tfrac{1}{4}a^4\,d\phi = \tfrac{1}{2}\pi Ka^4\left[-\tfrac{1}{4}\cos 2\phi\right]_0^{\pi/2} = \tfrac{\pi}{4}Ka^4.$

By the symmetry of the problem $M_{xz}=M_{yz}=0$, and

$M_{xy}=\int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^4\cos^2\phi\sin\phi\,d\rho\,d\phi\,d\theta = \tfrac{2}{5}\pi Ka^5\int_0^{\pi/2}\cos^2\phi\sin\phi\,d\phi = \tfrac{2}{5}\pi Ka^5\left[-\tfrac{1}{3}\cos^3\theta\right]_0^{\pi/2} = \tfrac{2}{15}\pi Ka^5.$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(0,0,\tfrac{8}{15}a\right)$.