Math 203 Sec AA Midterm Test 15 October 2003

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Instructions:	*Please answer all 5 questions, which carry equal marks. Explain your work clearly. Calculators are* **not** *permitted. The time allowed is 1 hour.*

1. Suppose $f(x) = 1 + x^2$ and $g(x) = \sin(3x - 1)$. Find the functions $f \circ g$, $g \circ f$, and $f \circ f$. [Recall *composition* $(f \circ g)(x) = f(g(x))$]

2. Find the inverse function $f^{-1}(x)$ of the function $y = f(x) = \sinh(x)$ and specify its domain. What is the value of $f^{-1}(1)$? [HINTS (i) $\sinh(x) = (e^x - e^{-x})/2$; (ii) it might help to set $z = e^x$ and solve first for z in terms of y.]

3. Find dy/dx if
   (a) $y = \ln(\cos(1 + x^3))$; (b) $y = (e^x - \cos(x))^5/(\tan^{-1}(x^2))^9$.

4. Find the equation of the tangent line which, at the point $(x, y) = (2, 1)$, touches the curve C defined implicitly by the equation $y^3x - (x+1)^2y + x^3 = 1$.

5. Consider the function $y = f(x) = \cos(x)e^{-x}$. If $y' = f'(x)$, and $y'' = f''(x)$, (a) find y'' and (b) show that $y'' + 2y' + 2y = 0$.

Solutions

1. $(f \circ g)(x) = 1 + (\sin(3x-1))^2$, $(g \circ f)(x) = \sin(x^2+2)$, and $(f \circ f)(x) = x^4 + 2x^2 + 2$.

2. We have $y = \sinh(x) = (e^x - e^{-x})/2 = (z - 1/z)/2$. Thus in order to find (first) z we must solve the quadratic equation $z^2 - 2yz - 1 = 0$. Using 'the formula', we find $z = y \pm \sqrt{1+y^2}$. Since $z = e^x > 0$, we must choose the $+$ sign and therefore $z = e^x = y + \sqrt{1+y^2}$. By taking logs, we find $x = f^{-1}(y) = \ln(y + \sqrt{1+y^2})$. The domain is R. In particular we have $f^{-1}(1) = \ln(1+\sqrt{2})$.

3.
   (a) $y' = -3x^2 \tan(1+x^3)$.

   (b) By logarithmic differentiation (or by the quotient rule) we get

$$y' = \frac{(e^x - \cos(x))^5}{(\tan^{-1}(x^2))^9} \left[\frac{5(e^x + \sin(x))}{e^x - \cos(x)} - \frac{36x}{(1+x^4)\tan^{-1}(x^2)} \right].$$

4. First we check that $P = (2,1)$ indeed satisfies the given equation. Then, by differentiation of the equation we have

$$3y'y^2x + y^3 - 2(x+1)y - (x+1)^2y' + 3x^2 = 0.$$

   By subsituting $x = 2, y = 1$ in this equation we can solve for y' to find $y' = 7/3$. Hence the equation of the tangent line which touches the curve at $(2,1)$ becomes $y - 1 = (7/3)(x-2)$, or $y = (7/3)x - 11/3$.

5. By differentiating we find $y' = -e^x(\cos(x) + \sin(x))$, and $y'' = 2\sin(x)e^x$. Substitution of these functions in the given 'differential equation' then demonstrates that $y'' + 2y' + 2y = 0$, as claimed.